Debye Sources and the Numerical Solution of the Time Harmonic Maxwell Equations

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Abstract

In this paper, we develop a new representation for outgoing solutions to the time-harmonic Maxwell equations in unbounded domains in $\mathbb{R}^3$. This representation leads to a Fredholm integral equation of the second kind for solving the problem of scattering from a perfect conductor, which does not suffer from spurious resonances or low-frequency breakdown, although it requires the inversion of the scalar surface Laplacian on the domain boundary. In the course of our analysis, we give a new proof of the existence of nontrivial families of time-harmonic solutions with vanishing normal components that arise when the boundary of the domain is not simply connected. We refer to these as k-Neumann fields, since they generalize, to nonzero wave numbers, the classical harmonic Neumann fields. The existence of k-Neumann fields was established earlier by Kress.

Introduction

Electromagnetic wave propagation in a uniform, nonconducting, isotropic medium in $\mathbb{R}^3$ is described by the Maxwell equations

$$\begin{aligned} \nabla \times \mathcal{E}(\boldsymbol{x}, t) &= -\mu \frac{\partial \mathcal{H}}{\partial t}, & \nabla \times \mathcal{H}(\boldsymbol{x}, t) &= \epsilon \frac{\partial \mathcal{E}}{\partial t}, \\ \nabla \cdot \mathcal{E}(\boldsymbol{x}, t) &= 0, & \nabla \cdot \mathcal{H}(\boldsymbol{x}, t) &= 0, \end{aligned} \tag{0.1}$$

where $\mathcal{E}$ and $\mathcal{H}$ denote the electric and magnetic fields, respectively, and ϵ and μ are the electrical permittivity and magnetic permeability of the medium. We restrict our attention to the time harmonic case and write

$$\mathcal{E}(\boldsymbol{x}, t) = \Re\left\{\frac{\boldsymbol{E}^{\text{tot}}(\boldsymbol{x})}{\sqrt{\epsilon}} e^{-i\omega t}\right\} \quad \text{and} \quad \mathcal{H}(\boldsymbol{x}, t) = \Re\left\{\frac{\boldsymbol{H}^{\text{tot}}(\boldsymbol{x})}{\sqrt{\mu}} e^{-i\omega t}\right\}. \tag{0.2}$$

The superscript is used to emphasize that $\boldsymbol{E}^{\text{tot}}$ and $\boldsymbol{H}^{\text{tot}}$ define the *total* electric and magnetic fields, respectively. In electromagnetic scattering, they are generally

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written as a sum

$$\begin{aligned} \boldsymbol{E}^{\text{tot}}(\boldsymbol{x}) &= \boldsymbol{E}^{\text{in}}(\boldsymbol{x}) + \boldsymbol{E}(\boldsymbol{x}), \\ \boldsymbol{H}^{\text{tot}}(\boldsymbol{x}) &= \boldsymbol{H}^{\text{in}}(\boldsymbol{x}) + \boldsymbol{H}(\boldsymbol{x}), \end{aligned} \tag{0.3}$$

where $\{\boldsymbol{E}^{\text{in}}, \boldsymbol{H}^{\text{in}}\}$ describe a known incident field and $\{\boldsymbol{E}, \boldsymbol{H}\}$ denote the scattered field of interest. With the scaling in (0.2) and $k = \sqrt{\epsilon\mu}\,\omega$, the Maxwell equations take the simpler form

$$\begin{aligned} \nabla \times \boldsymbol{H}^{\text{tot}} &= -ik\boldsymbol{E}^{\text{tot}} \\ \nabla \times \boldsymbol{E}^{\text{tot}} &= ik\boldsymbol{H}^{\text{tot}}. \end{aligned} \tag{0.4}$$

We are particularly interested in the problem of scattering from a perfect conductor in an exterior region, which we denote by Ω. For a perfect conductor [15, 24], the conditions to be enforced on Γ, the boundary of Ω, are

$$\begin{aligned} \boldsymbol{n} \times \boldsymbol{E}^{\text{tot}} &= 0, \\ \boldsymbol{n} \cdot \boldsymbol{H}^{\text{tot}} &= 0. \end{aligned} \tag{0.5}$$

The scattered field is assumed to satisfy the Silver-Müller radiation condition:

$$\lim_{r\to\infty} \left(\boldsymbol{H} \times \tfrac{\boldsymbol{r}}{r} - \boldsymbol{E}\right) = o\left(\tfrac{1}{r}\right). \tag{0.6}$$

This problem has been studied rather intensively for many decades, and we do not seek to review the literature here except to observe that there are two distinct approaches in widespread use. When the scatterer is a sphere, a simple and elegant theory exists due to Lorenz, Debye, and Mie [5, 12, 18, 21]. It is based on two scalar potentials (generally called Debye potentials) and the mathematical machinery of vector spherical harmonics. In particular, one represents $\boldsymbol{E}$ and $\boldsymbol{H}$ as

$$\begin{aligned} \boldsymbol{E} &= \nabla \times \nabla \times (\boldsymbol{r}v) + ik\nabla \times (\boldsymbol{r}u), \\ \boldsymbol{H} &= \nabla \times \nabla \times (\boldsymbol{r}u) - ik\nabla \times (\boldsymbol{r}v), \end{aligned} \tag{0.7}$$

where the Debye potentials u and v satisfy the scalar Helmholtz equation

$$\Delta u + k^2 u = 0, \quad \Delta v + k^2 v = 0,$$

with Helmholtz parameter (wave number) $k^2 = \omega^2\epsilon\mu$.

Salient features of this approach are that (1) the boundary value problem

$$\boldsymbol{n} \times \boldsymbol{E} = -\boldsymbol{n} \times \boldsymbol{E}^{\text{in}}, \tag{0.8}$$

$$\boldsymbol{n} \cdot \boldsymbol{H} = -\boldsymbol{n} \cdot \boldsymbol{H}^{\text{in}}, \tag{0.9}$$

is uniquely solvable for any k with nonnegative imaginary part, and (2) as $k \to 0$ ($\omega \to 0$), the electric and magnetic fields uncouple gracefully. In the static limit, $\boldsymbol{E}$ is due to the scalar potential v alone, which is, in turn, determined by the boundary data $-\boldsymbol{n} \times \boldsymbol{E}^{\text{in}}$. Likewise, $\boldsymbol{H}$ is due to the scalar potential u alone, which is determined by the boundary data $-\boldsymbol{n} \cdot \boldsymbol{H}^{\text{in}}$.

For regions of arbitrary shape, on the other hand, integral formulations of the Maxwell equations are generally based on the classical vector and scalar potentials (in the Lorenz gauge):

$$\boldsymbol{E} = ik\boldsymbol{A} - \nabla\phi, \tag{0.10}$$

$$\boldsymbol{H} = \nabla \times \boldsymbol{A}, \tag{0.11}$$

where

$$\boldsymbol{A}(\boldsymbol{x}) = \int_\Gamma g_k(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{J}(\boldsymbol{y})dA_{\boldsymbol{y}},$$

$$\phi(\boldsymbol{x}) = \frac{1}{ik}\int_\Gamma g_k(\boldsymbol{x}-\boldsymbol{y})(\nabla_\Gamma \cdot \boldsymbol{J})(\boldsymbol{y})dA_{\boldsymbol{y}},$$

with

$$g_k(\boldsymbol{x}) = \frac{e^{ik|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}.$$

Here $\boldsymbol{J}$ is a surface current (a tangential vector field) and $\nabla_\Gamma \cdot \boldsymbol{J}$ denotes its surface divergence.

Maue [19] proposed the electric field integral equation (EFIE) for the unknown current $\boldsymbol{J}$ by enforcing the condition (0.8) using the representation (0.10). Because of the $\nabla\phi$ term, however, the result is a hypersingular equation. Maue also proposed the magnetic field integral equation (MFIE), based on (0.11). The boundary condition for $\boldsymbol{H}$ can be derived from the Maxwell equations and an appropriate limiting process on the surface of a perfect conductor [15, 24],

$$\boldsymbol{J} = \boldsymbol{n} \times \boldsymbol{H}^{\text{in}} + \boldsymbol{n} \times \boldsymbol{H}, \tag{0.12}$$

where $\boldsymbol{n}$ points into Ω. Enforcing this condition for the unknown current $\boldsymbol{J}$ yields the MFIE, a second-kind Fredholm equation.

Unfortunately, both the MFIE and the EFIE have spurious resonances; that is, there exists a countable set of frequencies $\{k_j\} \subset \mathbb{R}$ for which the integral equations are not invertible. As the $\{k_j\}$ are the eigenvalues of a self-adjoint, elliptic boundary value problem on the bounded complement of Ω, they are often referred to as *interior resonances*. Below the smallest such k_j, the MFIE is well-conditioned, at least for simply connected domains. In the multiply connected case, the MFIE has a nontrivial null space in the static limit, carefully analyzed in the recent paper [11] and discussed in a slightly different form in [4]. This lack of invertibility has significant consequences on the conditioning of the MFIE at very low frequencies. Spurious resonances and the presence of a nontrivial null space in the static limit, however, are only one difficulty. A second problem stems from the representation of the electric field itself. Unlike the Debye representation, the electric field does not uncouple naturally from the magnetic field as $k \to 0$. Note that in (0.10), $\boldsymbol{E}$ involves one term of order k and one term of order k^{-1}. This results in what is referred to as "low-frequency breakdown" [38]. While low-frequency breakdown is a more transparent problem in the context of the EFIE, the MFIE is not immune [37]. Knowing the current $\boldsymbol{J}$ is sufficient for computing $\boldsymbol{H}$ but not the electric field. The normal component of $\boldsymbol{E}$, for example, is determined by the electric charge:

$$\boldsymbol{n} \cdot \boldsymbol{E} = \rho = \frac{\nabla_\Gamma \cdot \boldsymbol{J}}{ik}. \tag{0.13}$$

As $k \to 0$, accuracy degrades dramatically—a phenomenon called "catastrophic cancellation" in numerical analysis.

This state of affairs is both odd and unsatisfactory. For the exterior of a sphere, there is a simple, clean representation of the solution based on two scalar unknowns that results in a diagonal linear system. It has no spurious resonances and, at zero frequency, decouples naturally (with no loss of precision) into magnetostatic and electrostatic problems. The standard integral equation approaches available for general geometries do not reduce to a Debye-like formalism when restricted to a sphere. Instead, a sequence of modifications have been introduced to address the three problems discussed above: the existence of spurious resonances, the lack of a second kind of integral equation valid for all frequencies, and the loss of accuracy due to low-frequency breakdown.

An important step in addressing the first problem was the introduction in the 1970s of the combined field integral equation (CFIE) [22, 27]. The CFIE avoids spurious resonances by taking a complex linear combination of the EFIE and the MFIE, both of which involve the surface current as the unknown. It is not a Fredholm equation of the second kind, however, and still suffers from low-frequency breakdown. One alternative approach, due to Yaghjian [35], involves augmenting the MFIE with condition (0.9) or the EFIE with condition (0.13). He showed that (for geometries other than the sphere) the augmented equations yield unique solutions at all frequencies. Of the many formulations that have been introduced to overcome spurious resonances, variants of the CFIE have emerged as the most frequently used in practice.

For the second problem, the principal issue is that of overcoming the hypersingular behavior of the CFIE. One solution is to introduce electric charge ρ as an additional variable [29]. In this approach, one defines the scalar potential ϕ by

$$\phi(\boldsymbol{x}) = \int_\Gamma g_k(\boldsymbol{x} - \boldsymbol{y})\rho(\boldsymbol{y})dA_{\boldsymbol{y}} \tag{0.14}$$

and imposes the continuity condition

$$\nabla \cdot \boldsymbol{J} = ik\rho. \tag{0.15}$$

While the hypersingular term is avoided, one must solve a Fredholm integral equation subject to a differential-algebraic constraint (0.15). Over the last several years, promising approaches have been developed based on the construction of preconditioners. Christiansen and Nédélec [7] designed effective strategies for the EFIE based on Calderon formulas and the Helmholtz decomposition. Colton and Kress developed a regularized CFIE to prove uniqueness for the exterior Maxwell system [9, theorem 6.19], which can be interpreted as a "right" preconditioner in linear algebra terminology. A similar approach has recently been proposed and shown to work well in numerical experiments by Bruno et al. [6].

Adams [1] and Contopanagos et al. [10] made use of the fact that the EFIE operator serves as its own preconditioner; more precisely, the composition of the hypersingular operator with itself equals the sum of the identity operator and a compact operator. A combined field integral equation using this preconditioned EFIE is both resonance-free and takes the form of a Fredholm equation of the second kind. Preconditioners have also been designed through the use of high-frequency asymptotics [2]. Unfortunately, the implementation of these schemes can be rather involved on arbitrary surfaces and, like the MFIE, they still suffer from a form of low-frequency breakdown in the evaluation of $\boldsymbol{E}$ once the integral equation has been solved.

Finally, the third problem—namely, the low-frequency breakdown of the integral equations—has generally been handled through the use of specialized basis functions in the discretization of the current, such as the "loop and tree" method of [33, 34]. This is a kind of discrete surface Helmholtz decomposition of $\boldsymbol{J}$. As the frequency goes to 0, the irrotational and solenoidal discretization elements become uncoupled, avoiding the scaling problem that causes loss of precision.

We have chosen to investigate a rather different line of thought, motivated largely by the desire to extend the Debye potentials to surfaces of arbitrary shape. In essence, the Lorenz-Debye-Mie approach is based on expanding the potentials u and v from (0.7) as

$$\begin{aligned} v(r,\theta,\phi) &= \sum_{n,m} a_{n,m} h_n^{(1)}(kr) Y_n^m(\theta,\phi), \\ u(r,\theta,\phi) &= \sum_{n,m} b_{n,m} h_n^{(1)}(kr) Y_n^m(\theta,\phi), \end{aligned}$$

where $h_n^{(1)}(x)$ is the spherical Hankel function of order n, and $Y_n^m(\theta,\phi)$ is the usual spherical harmonic of order n and degree m. This separation-of-variables approach is clearly not suitable in general. From a mathematical viewpoint, it works because of the close connection between the Laplacian in $\mathbb{R}^3$ and the surface Laplace-Beltrami operator on the sphere. It is also worth noting that the Lorenz-Debye-Mie approach is not equivalent to a Fredholm equation of the second kind. It is invertible and resonance free and behaves properly at low frequencies, but it is hypersingular. Numerical difficulties are avoided simply because it is in diagonal form.

The features of the Debye potentials that we wish to retain are their symmetry and the fact that, at zero frequency, the system uncouples into separate electrostatic and magnetostatic problems. For symmetry, we begin by using both potentials

$(\boldsymbol{A}, \phi)$ and "antipotentials" $(\boldsymbol{A}_m, \phi_m)$ as a formal representation of the electromagnetic fields [24]:

$$\boldsymbol{E} = ik\boldsymbol{A} - \nabla\phi - \nabla \times \boldsymbol{A}_m, \tag{0.16}$$

$$\boldsymbol{H} = \nabla \times \boldsymbol{A} + ik\boldsymbol{A}_m - \nabla\phi_m, \tag{0.17}$$

where

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{x}) &= \int_\Gamma g_k(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{j}(\boldsymbol{y})dA_{\boldsymbol{y}}, \\ \phi(\boldsymbol{x}) &= \int_\Gamma g_k(\boldsymbol{x}-\boldsymbol{y})r(\boldsymbol{y})dA_{\boldsymbol{y}}, \\ \boldsymbol{A}_m(\boldsymbol{x}) &= \int_\Gamma g_k(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{m}(\boldsymbol{y})dA_{\boldsymbol{y}}, \\ \phi_m(\boldsymbol{x}) &= \int_\Gamma g_k(\boldsymbol{x}-\boldsymbol{y})q(\boldsymbol{y})dA_{\boldsymbol{y}}, \end{aligned} \tag{0.18}$$

together with the continuity conditions

$$\nabla_\Gamma \cdot \boldsymbol{j} = ikr, \quad \nabla_\Gamma \cdot \boldsymbol{m} = ikq. \tag{0.19}$$

Such a symmetric formulation is commonly used for scattering from a dielectric. For the perfect conductor, it underlies the combined source integral equation method (CSIE) [20], where $\boldsymbol{j}$ and $\boldsymbol{m}$ are both assumed to be derived from a "parent" current distribution $\tilde{\boldsymbol{j}}$:

$$\boldsymbol{j} = \alpha\tilde{\boldsymbol{j}}, \quad \boldsymbol{m} = (1-\alpha)\boldsymbol{n} \times \tilde{\boldsymbol{j}},$$

for some parameter α. More precisely, the CSIE is derived using (0.16) with the vector unknown $\tilde{\boldsymbol{j}}$ and enforcing the condition (0.8). Like the CFIE, it is a resonance-free but hypersingular equation. It is important to recognize that, in this construction, the unknowns $\boldsymbol{j}$, r, $\boldsymbol{m}$, and q are no longer physical quantities; $\boldsymbol{j}$ and r correspond to *fictitious* electric current and electric charge, while $\boldsymbol{m}$ and q correspond to *fictitious* magnetic current and magnetic charge. Perfect conductors do not support the latter. If the "physical" current supported on the surface Γ is desired, it must be computed in a second step. From (0.12), for example, we have $\boldsymbol{J} = \boldsymbol{n} \times (\boldsymbol{H}^{\text{in}} + \boldsymbol{H})$. This is *not* the unknown $\boldsymbol{j}$.

The second and critical feature of our method is that we use r and q as unknowns and *construct* $\boldsymbol{j}$ and $\boldsymbol{m}$ from them in such a way that the continuity conditions (0.19) are automatically satisfied. In particular, for simply connected domains, we let

$$\boldsymbol{j} = \nabla_\Gamma\Psi + \boldsymbol{n} \times \nabla_\Gamma\Psi_m, \quad \boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{j}, \tag{0.20}$$

where

$$\begin{aligned}\Delta_\Gamma \Psi &\equiv \nabla_\Gamma^2 \Psi = ikr,\\ \Delta_\Gamma \Psi_m &= \nabla_\Gamma^2 \Psi_m = -ikq.\end{aligned} \tag{0.21}$$

We refer to Δ_Γ as the surface Laplacian or Laplace-Beltrami operator. (In geometry, this name is usually applied to $-\Delta_\Gamma$ so that it is a nonnegative operator, but we use the convention above consistently.) In any case, we obtain the Helmholtz decomposition of the currents on the surface by construction. This avoids the obvious cause of low-frequency breakdown, since we never compute the $O(1)$ quantities r and q from the $O(k)$ quantities $\boldsymbol{j}$ and $\boldsymbol{m}$ with its attendant loss of accuracy.

An obvious drawback of our approach, of course, is that it requires the inversion of a partial differential equation on the surface of the scatterer to compute Ψ and Ψ_m. It is interesting to note that Scharstein proposed an investigation along these lines some years ago [28], using only the electric current $\boldsymbol{J}$, but a detailed investigation of the theory was not carried out.

We show below that our representation yields a second-kind integral equation for r and q. We remark that an equation is of the "second kind" if it can be written in the form $(\mathrm{Id} + K)x = y$, where K is a "compact" operator. In this paper, the compact operators are all classical pseudodifferential operators of negative, integral order and are therefore compact as maps from $H^s(\Gamma)$ to itself for any real number s.

In the simply connected case, this system of equations has a unique solution for all frequencies with nonnegative imaginary part. Furthermore, it behaves gracefully in the low-frequency limit, uncoupling into an electrostatic problem involving r and a magnetostatic problem involving q. Because of the connection with the Debye theory, we refer to r and q as *generalized Debye sources*. In the multiply connected case, we augment the system of integral equations with a finite number of additional equations to again obtain a Fredholm system of the second kind without interior resonances. The behavior as k tends to 0 is more complicated. Our results in this paper on the low-frequency limit are not definitive in the multiply connected case. To avoid low-frequency breakdown, the integral equations have to be augmented with a different finite set of equations. We will return to this problem in a subsequent publication.

1 Geometric Preliminaries

DEFINITION 1.1 Let D denote a bounded (not necessarily connected) region in $\mathbb{R}^3$ and let Ω denote the unbounded component of $\mathbb{R}^3 \setminus \overline{D}$. We refer to Ω as the *exterior region* and to its boundary as Γ. We assume, without loss of generality, that $\mathbb{R}^3 \setminus D$ has no bounded components (that is, holes within the interior of D).

Using standard topological terminology, let us assume that D is multiply connected with genus g. Then there exist surfaces $S_1, \dots, S_g$ in D such that $D \setminus \bigcup_{j=1}^{g} S_j$ is simply connected and surfaces $T_1, \dots, T_g$ in $\mathbb{R}^3 \setminus D$ such that $\mathbb{R}^3 \setminus$

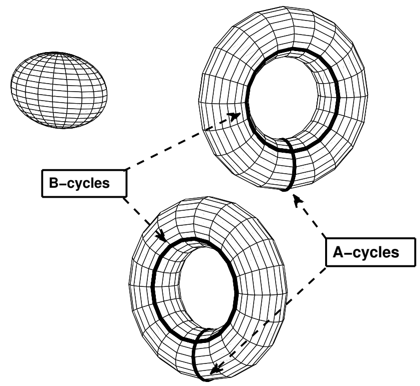


FIGURE 1.1. A multiply connected domain D, consisting of three components (two of genus 1 and one of genus 0), with exterior Ω. Cutting along the surfaces bounded by the "A-cycle" from D makes it simply connected. Adding the surfaces T_j bounded by the B-cycles makes $\Omega \setminus D$ simply connected.

$D \setminus \bigcup_{j=1}^{g} T_j$ is simply connected. We denote by A_j the boundary of S_j and by B_j the boundary of T_j (see Figure 1.1).

Remark 1.2. We refer to the curves $\{A_j : j = 1, \dots, g\}$ as *A-cycles*. (They form a basis for the first homology group of $\mathbb{R}^3 \setminus D$.) We refer to the curves $\{B_j : j = 1, \dots, g\}$ as *B-cycles*. (They form a basis for the first homology group of D.)

DEFINITION 1.3 Let Γ_j denote a component of the boundary Γ. If

$$\int_{\Gamma_j} f(\boldsymbol{x}) dA(\boldsymbol{x}) = 0,$$

we refer to it as having *mean zero* on that component. We denote by $\mathcal{M}_{\Gamma,0}$ the set of pairs of functions (f, g) on Γ with mean zero on every component.

LEMMA 1.4 (Mean Zero Condition) *Let (r, q) be generalized Debye sources defined on a boundary Γ. Then $(r, q) \in \mathcal{M}_{\Gamma,0}$.*

This is proven in Section 6.2 (see equations (6.36) and (6.37)).

In multiply connected domains, the Helmholtz decomposition (0.20) is incomplete. From Hodge theory, however, we can write a surface vector field $\boldsymbol{j}$ as an orthogonal decomposition (in L_2), the Hodge-Helmholtz decomposition:

$$\begin{aligned} \boldsymbol{j} &= \boldsymbol{j}_R + \boldsymbol{j}_H, \\ \boldsymbol{j}_R &= \nabla_\Gamma \Psi + \boldsymbol{n} \times \nabla_\Gamma \Psi_m, \end{aligned} \tag{1.1}$$

for some Ψ, Ψ_m, where $\boldsymbol{j}_H$ satisfies

$$\nabla_\Gamma \cdot \boldsymbol{j}_H = 0, \quad \nabla_\Gamma \cdot (\boldsymbol{n} \times \boldsymbol{j}_H) = 0.$$

Such vector fields are called *harmonic vector fields* and are the metric duals to harmonic 1-forms; see [31]. We let

$$\boldsymbol{m}_R = \boldsymbol{n} \times \boldsymbol{j}_R, \quad \boldsymbol{m}_H = \boldsymbol{n} \times \boldsymbol{j}_H,$$

and (as before)

$$\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{j}.$$

Harmonic vector fields arise, in essence, because the Laplace-Beltrami operator on vector fields

$$\Delta_\Gamma^1 \boldsymbol{j} \equiv \nabla_\Gamma \nabla_\Gamma \cdot \boldsymbol{j} - \boldsymbol{n} \times \nabla_\Gamma \nabla_\Gamma \cdot (\boldsymbol{n} \times \boldsymbol{j}) \tag{1.2}$$

has a nontrivial null space on multiply connected surfaces. The dimension of the null space of Δ_Γ^1 is equal to twice the genus g of the surface. We may therefore choose harmonic vector fields $\{\boldsymbol{j}_H^l : l = 1, \dots, 2g\}$ that form an orthogonal basis with respect to $L^2(\Gamma)$ for this null space. In the multiply connected case, we can then define the harmonic components of $\boldsymbol{j}$ and $\boldsymbol{m}$ by

$$\boldsymbol{j}_H = \sum_{l=1}^{2g} c_l \boldsymbol{j}_H^l, \quad \boldsymbol{m}_H = \boldsymbol{n} \times \boldsymbol{j}_H. \tag{1.3}$$

The space of harmonic vector fields is denoted $\mathcal{H}^1(\Gamma)$.

Given that the Laplace-Beltrami operator is not invertible, one must be careful in defining Ψ and Ψ_m. From Hodge theory, however, we know that it is invertible as a map from the space of mean zero functions $\mathcal{M}_{\Gamma,0}$ to itself. We denote by R_0 the *partial inverse* of Δ_Γ acting on this space. Thus, we replace (0.21) with

$$\Psi = ikR_0 r, \quad \Psi_m = -ikR_0 q, \tag{1.4}$$

where r and q are the generalized Debye sources.

EXAMPLE 1.5 Consider a torus in $\mathbb{R}^3$ parametrized by

$$\boldsymbol{x}(s,t) = [(R + r\cos t)\cos s, (R + r\cos t)\sin s, r\sin t]$$

with the z-axis as the axis of symmetry. A straightforward calculation shows that

$$\boldsymbol{j}_H^1 = \frac{1}{(R + r\cos t)^2} \frac{\partial \boldsymbol{x}(s,t)}{\partial s} \quad \text{and} \quad \boldsymbol{j}_H^2 = \boldsymbol{n} \times \boldsymbol{j}_H^1$$

are both harmonic vector fields. Since the genus of a torus is 1, they form a basis for the two-dimensional space of harmonic vector fields on the surface.

Remark 1.6. Much of the formal analysis in this paper is simplified through the use of differential forms and homology theory. In order to be accessible to a broader audience, however, we state the main results using the notation of vector calculus and defer most proofs to Sections 6 and 7, where we do make use of the language and power of this theory.

2 Uniqueness Theorems for Exterior Electromagnetic Fields

Let us denote by $\overline{\mathbb{C}_+}$ the closed upper half-plane:

$$\overline{\mathbb{C}_+} = \{z \in \mathbb{C} : \Im z \geq 0\}.$$

DEFINITION 2.1 A solution to the time-harmonic Maxwell equations in Ω that satisfies the Silver-Müller radiation condition is referred to as an *outgoing solution*.

That an outgoing solution to THME(k) is determined by either the tangential components of the electric or magnetic fields is classical [8, 9, 23]:

THEOREM 2.2 *Suppose that $(\boldsymbol{E}, \boldsymbol{H})$ is an outgoing solution to the THME(k) in an exterior region Ω for nonzero $k \in \overline{\mathbb{C}_+}$. If either $\boldsymbol{n} \times \boldsymbol{E}$ or $\boldsymbol{n} \times \boldsymbol{H}$ vanishes on Γ, then the solution is identically zero in Ω.*

Although we will eventually address the problem of scattering from a perfect conductor ($\boldsymbol{n} \times \boldsymbol{E} = 0$), we turn our attention for the moment to the Maxwell equations in exterior domains with normal components specified on the boundary. While this is not a standard physical boundary value problem, there is prior work on uniqueness, and it is a natural starting point for the analysis of symmetric representations of the fields.

THEOREM 2.3 (Yee [36]) *Let $(\boldsymbol{E}, \boldsymbol{H})$ be an outgoing solution to the THME(k) in an exterior region Ω for nonzero $k \in \overline{\mathbb{C}_+}$. Suppose Γ is simply connected and that*

$$\boldsymbol{n} \cdot \boldsymbol{E} \restriction_{\Gamma} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{H} \restriction_{\Gamma} = 0.$$

Then $\boldsymbol{E} = \boldsymbol{H} = 0$ in Ω.

When the boundary has nontrivial topology, a rather subtle argument shows that, in general, this is not true. In particular, if the sum of the genera of the boundary components of the exterior domain is $g > 0$, then for all frequencies with nonnegative imaginary part, there is a $2g$-dimensional space of outgoing solutions to the THME with vanishing normal components. The existence of these fields was proven by Kress [17] (Theorem 2.5 below).

Remark 2.4. In the static (harmonic) case this fact has been known for decades [32]. More precisely, at $k = 0$, the THME separate into the system

$$\nabla \cdot \boldsymbol{E} = 0,\ \nabla \times \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{H} = 0,\ \nabla \times \boldsymbol{H} = 0,$$

solutions to which are called harmonic vector fields (if they decay at infinity). When their normal components vanish on the boundary, they are called harmonic Neumann fields. If their tangential components vanish, they are called harmonic Dirichlet fields.

THEOREM 2.5 (Kress [17]) *Suppose that $(\boldsymbol{E}, \boldsymbol{H})$ is an outgoing solution to the THME(k) in the exterior region Ω for nonzero $k \in \overline{\mathbb{C}_+}$. If every component of the boundary Γ is simply connected, then the solution is determined by the normal* *components* $\boldsymbol{n}\cdot\boldsymbol{E}\restriction_\Gamma$ *and* $\boldsymbol{n}\cdot\boldsymbol{H}\restriction_\Gamma$. *If the sum of the genera of the components of* Γ *equals* $g>0$, *then there is a subspace of outgoing solutions to THME*(k) *with*

$$\boldsymbol{n}\cdot\boldsymbol{E}\restriction_\Gamma = 0 \quad \textit{and} \quad \boldsymbol{n}\cdot\boldsymbol{H}\restriction_\Gamma = 0 \tag{2.1}$$

of dimension exactly $2g$.

LEMMA 2.6 (Kress [17]) *Let* $(\boldsymbol{E},\boldsymbol{H})$ *be a solution to the THME*(k) *for nonzero* $k\in\overline{\mathbb{C}_+}$ *in a region* Ω *with a smooth bounded boundary* Γ. *Then the normal components* $(\boldsymbol{n}\cdot\boldsymbol{E},\boldsymbol{n}\cdot\boldsymbol{H})$ *lie in* $\mathcal{M}_{\Gamma,0}$.

COROLLARY 2.7 (Kress [17]) *Let* $(\boldsymbol{E},\boldsymbol{H})$ *be an outgoing solution to the THME*(k) *with* $k\in\overline{\mathbb{C}_+}$ *and satisfying*

$$\boldsymbol{n}\cdot\boldsymbol{E}\restriction_\Gamma = f \quad \textit{and} \quad \boldsymbol{n}\cdot\boldsymbol{H}\restriction_\Gamma = h$$

where $f,h\in\mathcal{M}_{\Gamma,0}$ *are Hölder-continuous. Then* $(\boldsymbol{E},\boldsymbol{H})$ *are uniquely determined subject to the specification of their circulations on the* A*-cycles* (*see Figure* 1.1):

$$\int_{A_j}\boldsymbol{\tau}\cdot\boldsymbol{E}\,ds = p_j \quad \textit{and} \quad \int_{A_j}\boldsymbol{\tau}\cdot\boldsymbol{H}\,ds = q_j, \tag{2.2}$$

where $p_j,q_j\in\mathbb{C}$ *are given numbers.*

Remark 2.8. We call solutions to THME(k) that satisfy (2.1) k*-Neumann fields*, and denote the space of such solutions by $\mathcal{H}_k(\Omega)$. The conditions in Corollary 2.7 are familiar from the static (zero frequency) case, where g conditions must be specified for each of $\boldsymbol{E}$ and $\boldsymbol{H}$ separately, since the equations are uncoupled. For nonzero k, this symmetry is not required. We provide a different proof of existence in Theorem 7.4 and a somewhat more general analysis of uniqueness in Section 6.1.

Our representation also provides an effective means for computing the k-Neumann fields. These solutions are needed in solving the problem of scattering from a multiply connected perfect conductor.

First, however, we need to recall some classical facts about layer potentials.

3 Jump Relations and Boundary Values of the Potentials

In order to use the integral representations discussed above to solve boundary value problems, we need to find expressions for the restrictions of $\boldsymbol{E}$ and $\boldsymbol{H}$ to the boundary in terms of the various potentials. In this section we collect the relevant results. Recall that $\Gamma\hookrightarrow\mathbb{R}^3$ is a smooth, bounded surface (possibly disconnected). The unbounded component of $\mathbb{R}^3\setminus\Gamma$, which we have denoted by Ω, is referred to as the "+" side of the boundary. The domain D (the union of the bounded components) is referred to as the "–" side. We use $\boldsymbol{n}$ to denote the unit normal vector field along Γ, pointing into the unbounded component.

The relevant limits are given in the following lemma, proofs of which can be found, for example, in [8, 9, 23].

LEMMA 3.1 *Let $\boldsymbol{A}$ and ϕ denote the vector and scalar potentials in* (0.18)*, and let Γ be a smooth, bounded surface in $\mathbb{R}^3$. For $\boldsymbol{x}_0 \in \Gamma$, let $\boldsymbol{x} \to \boldsymbol{x}_0^{\pm}$ indicate the approach from Ω $(+)$ or D $(-)$, respectively, and let $\boldsymbol{n}_0$ denote the normal at $\boldsymbol{x}_0$, with $\frac{\partial}{\partial n_0}$ the normal derivative at $\boldsymbol{x}_0$. Then, for the scalar potential, we have*

$$\begin{aligned}
&\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm}} \boldsymbol{n}\cdot\nabla\phi_{\pm}(\boldsymbol{x}) = \mp\frac{1}{2}r(\boldsymbol{x}_0) + K_0[r](\boldsymbol{x}_0),\\
&\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm}} \boldsymbol{n}\times\nabla\phi_{\pm}(\boldsymbol{x}) = K_1[r](\boldsymbol{x}_0),
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
K_0[r](\boldsymbol{x}_0) &= \int_\Gamma \frac{\partial g_k}{\partial n_0}(\boldsymbol{x}_0-\boldsymbol{x})r(\boldsymbol{x})dA(\boldsymbol{x}),\\
K_1[r](\boldsymbol{x}_0) &= \boldsymbol{n}_0\times\nabla\int_\Gamma g_k(\boldsymbol{x}_0-\boldsymbol{x})r(\boldsymbol{x})dA(\boldsymbol{x}),
\end{aligned}$$

K_0 is an integral operator of order -1*, and K_1 is an integral operator of order* 0*, which is defined in a principal value sense.*

For the vector potential we have

$$\begin{aligned}
&\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm}} \boldsymbol{n}\cdot\boldsymbol{A}_{\pm}(\boldsymbol{x}) = K_{2,n}[\boldsymbol{j}](\boldsymbol{x}_0),\\
&\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm}} \boldsymbol{n}\times\boldsymbol{A}_{\pm}(\boldsymbol{x}) = K_{2,t}[r](\boldsymbol{x}_0),
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
K_{2,n}[\boldsymbol{j}](\boldsymbol{x}_0) &= \int_\Gamma g_k(\boldsymbol{x}_0-\boldsymbol{x})(\boldsymbol{n}_0\cdot\boldsymbol{j}(\boldsymbol{x}))dA(\boldsymbol{x}),\\
K_{2,t}[\boldsymbol{j}](\boldsymbol{x}_0) &= \int_\Gamma g_k(\boldsymbol{x}_0-\boldsymbol{x})(\boldsymbol{n}_0\times\boldsymbol{j}(\boldsymbol{x}))dA(\boldsymbol{x}).
\end{aligned}$$

$K_{2,n}$ and $K_{2,t}$ are both integral operators of order -1.

The vector potential also satisfies

$$\begin{aligned}
&\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm}} \boldsymbol{n}\cdot\nabla\times\boldsymbol{A}_{\pm}(\boldsymbol{x}) = K_3[\boldsymbol{j}](\boldsymbol{x}_0),\\
&\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm}} \boldsymbol{n}\times\nabla\times\boldsymbol{A}_{\pm}(\boldsymbol{x}) = \pm\frac{1}{2}\boldsymbol{j}(\boldsymbol{x}_0) + K_4[\boldsymbol{j}](\boldsymbol{x}_0),
\end{aligned} \tag{3.3}$$

where

$$K_3[\boldsymbol{j}](\boldsymbol{x}_0) = \int\limits_\Gamma \nabla g_k(\boldsymbol{x}_0 - \boldsymbol{x}) \cdot (\boldsymbol{j}(\boldsymbol{x}) \times \boldsymbol{n}_0)\, dA(\boldsymbol{x}),$$

$$K_4[\boldsymbol{j}](\boldsymbol{x}_0) = \int\limits_\Gamma \left[\nabla g_k(\boldsymbol{x}_0 - \boldsymbol{x})\,(\boldsymbol{j}(\boldsymbol{x}) \cdot \boldsymbol{n}_0) - \frac{\partial g_k}{\partial n_0}(\boldsymbol{x}_0 - \boldsymbol{x})\, \boldsymbol{j}(\boldsymbol{x})\right] dA(\boldsymbol{x}).$$

K_3 *is an integral operator of order* 0*.* K_4 *is an integral operator of order* -1*.*

PROOF: These results follow from classical potential theory, the observation that the kernels in $K_{2,n}$ and $K_{2,t}$ are weakly singular, and the fact that, at the singular point $\boldsymbol{x}_0 = \boldsymbol{x}$ in K_4, $\boldsymbol{n}_0$ is orthogonal to $\boldsymbol{j}(\boldsymbol{x})$). □

Remark 3.2. We have abused notation slightly in the preceding lemma. The operators are functions of the Helmholtz parameter k. When the explicit dependence is relevant, we occasionally write $K_0(k), K_1(k), \dots, K_4(k)$ instead.

The limits of the antipotentials $\boldsymbol{A}_m$ and ϕ_m are analogous. Recall, however, that we have chosen not to work with $\boldsymbol{j}$ and $\boldsymbol{m}$ as unknowns, but rather the generalized Debye sources r and q complemented by the harmonic vector fields. We compute $\boldsymbol{j}$ and $\boldsymbol{m}$ from

$$\boldsymbol{j} = \nabla_\Gamma \Psi + \nabla_\Gamma \times (\boldsymbol{n}\Psi_m) + \boldsymbol{j}_H, \quad \boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{j},$$

where Ψ and Ψ_m satisfy the Laplace-Beltrami equations (1.4) with r and q viewed as source data.

LEMMA 3.3 *The integral operators* K_2*,* K_3*, and* K_4 *are all of order* -1 *or* -2 *when viewed as operators acting on* r *and* q*. Hence they are compact as maps from* $H^s(\Gamma)$ *to itself for any real number* s*.*

PROOF: This follows immediately from Lemma 3.1 and the fact that $\boldsymbol{j}$ and $\boldsymbol{m}$ are of order -1 in terms of (r, q). □

From Lemma 3.1, we obtain the following jump relations:

COROLLARY 3.4 *Suppose that the fields* $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *are defined in terms of potentials and antipotentials. Then they satisfy*

$$\begin{aligned} \boldsymbol{n} \cdot (\boldsymbol{E}_+ - \boldsymbol{E}_-) &= r, & \boldsymbol{n} \times (\boldsymbol{E}_+ - \boldsymbol{E}_-) &= -\boldsymbol{m}, \\ \boldsymbol{n} \cdot (\boldsymbol{H}_+ - \boldsymbol{H}_-) &= q, & \boldsymbol{n} \times (\boldsymbol{H}_+ - \boldsymbol{H}_-) &= \boldsymbol{j}. \end{aligned} \tag{3.4}$$

4 Maxwell Equations with Normal Components Specified

We note that for $k \in \overline{\mathbb{C}_+}$ the fundamental solution

$$g_k(\boldsymbol{x}) = \frac{e^{ik|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|} \tag{4.1}$$

is outgoing; that is, it satisfies the Silver-Müller radiation condition. Thus, all of the corresponding potentials defined over bounded regions are outgoing as well.

THEOREM 4.1 *Let $\boldsymbol{E}$ and $\boldsymbol{H}$ be outgoing fields represented in terms of Debye sources (r,q) and currents $\boldsymbol{j}$ and $\boldsymbol{m}$. Then the limiting values of their normal components are given by the following integral representation for $\boldsymbol{x}_0 \in \Gamma$:*

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm}} \begin{pmatrix} \boldsymbol{E}\cdot\boldsymbol{n} \\ \boldsymbol{H}\cdot\boldsymbol{n} \end{pmatrix} = \begin{pmatrix} \pm\frac{1}{2}I - K_0 & 0 \\ 0 & \mp\frac{1}{2}I + K_0 \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix} + \begin{pmatrix} ikK_{2,n} & -K_3 \\ K_3 & ikK_{2,n} \end{pmatrix} \begin{pmatrix} \boldsymbol{j} \\ \boldsymbol{m} \end{pmatrix}, \tag{4.2}$$

where I denotes the identity operator. If we assume that $\boldsymbol{j} = \boldsymbol{j}_R(r,q,k)$ and $\boldsymbol{m} = \boldsymbol{n}\times\boldsymbol{j}$, then these are Fredholm integral operators of the second kind in the generalized Debye sources (r,q). As k tends to 0, this representation converges to

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm}} \begin{pmatrix} \boldsymbol{E}\cdot\boldsymbol{n} \\ \boldsymbol{H}\cdot\boldsymbol{n} \end{pmatrix} = \begin{pmatrix} \pm\frac{1}{2}I - K_0(0) & 0 \\ 0 & \mp\frac{1}{2}I + K_0(0) \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix}. \tag{4.3}$$

The equation $[-I + 2K_0(0)]f = h$ is uniquely solvable for all h, and the equation $[I + 2K_0(0)]f = h$ for all h of mean zero.

PROOF: The equations follow from the formulæ in the previous section. The solvability properties of $I \pm 2K_0(0)$ are classical and can be found in [8]. $\square$

DEFINITION 4.2 We let $\mathcal{N}^{\pm}(k)$ denote the operator on the right-hand side of (4.2), with $\mathcal{N}^{\pm}_{\boldsymbol{E}}(k)$ the first row and $\mathcal{N}^{\pm}_{\boldsymbol{H}}(k)$ the second.

If we seek to impose the boundary conditions $\boldsymbol{E}\cdot\boldsymbol{n}\restriction_{\Gamma} = f$ and $\boldsymbol{H}\cdot\boldsymbol{n}\restriction_{\Gamma} = h$, then we obtain the following system of equations, which is analytic in k:

$$\begin{pmatrix} \pm\frac{1}{2}I - K_0 & 0 \\ 0 & \mp\frac{1}{2}I + K_0 \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix} + \begin{pmatrix} ikK_{2,n} & -K_3 \\ K_3 & ikK_{2,n} \end{pmatrix} \begin{pmatrix} \boldsymbol{j} \\ \boldsymbol{m} \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}. \tag{4.4}$$

When $\boldsymbol{j} = \boldsymbol{j}_R(r,q,k)$ and $\boldsymbol{m} = \boldsymbol{n}\times\boldsymbol{j}$ are obtained from the Debye sources via (1.4), we denote the left-hand side of (4.4) by $\mathcal{N}^{\pm}(k)(r,q)$ and observe that it differs from $J_{\pm}(0)$ by an operator of negative order, where

$$J_{\pm}(k) = \begin{pmatrix} \pm\frac{1}{2}I - K_0(k) & 0 \\ 0 & \mp\frac{1}{2}I + K_0(k) \end{pmatrix}. \tag{4.5}$$

THEOREM 4.3 *Let $(f,h) \in \mathcal{M}_{\Gamma,0}$. For $k \notin E_+$, a discrete set in the complex plane, the equation*

$$\mathcal{N}^{+}(k)(r,q) = (f,h) \tag{4.6}$$

has a unique solution. The outgoing solution of the THME(k) defined by this data satisfies

$$\boldsymbol{E}_+\cdot\boldsymbol{n}\restriction_{\Gamma} = f \quad \boldsymbol{H}_+\cdot\boldsymbol{n}\restriction_{\Gamma} = h. \tag{4.7}$$

PROOF: We know from Theorem 4.1 that the operator $J_+(0)$ is invertible. By analytic Fredholm theory, therefore, we know that there is a discrete set $E_+ \in \mathbb{C}$ so that for $k \notin E_+, \mathcal{N}^+(k)$ is also invertible. The result now follows from the fact that $(r,q) \in \mathcal{M}_{\Gamma,0}$ (Lemma 1.4) and that $(f,h) \in \mathcal{M}_{\Gamma,0}$ (Lemma 2.6), so that

$$\mathcal{N}^+(k) : \mathcal{M}_{\Gamma,0} \to \mathcal{M}_{\Gamma,0}.$$

$\square$

COROLLARY 4.4 *When* Γ *is simply connected, the Fredholm equation* (4.6) *provides a unique solution to the THME*(k) *for any* $k \in \overline{\mathbb{C}_+}$.

PROOF: This follows immediately from Theorem 4.3 and the fact that E_+ lies in the complex lower half-plane, as follows from Corollary 7.2 in Section 7. $\square$

Remark 4.5. The problem of solving the THME with specified normal components was previously analyzed by Gülzow [14], who constructed a hypersingular integral equation method. Using generalized Debye sources instead leads to a well-conditioned integral equation of the second kind. In the multiply connected case, Theorem 4.3 shows that the problem of scattering with normal components specified is invertible if the solution is sought with zero projection onto the harmonic vector fields. The extra conditions in Kress' result (Corollary 2.7) force uniqueness on $\boldsymbol{j}_H$ by specifying g conditions on the circulation of $\boldsymbol{E}$ and g conditions on the circulation of $\boldsymbol{H}$, where g is the genus of Γ. In fact, any set of $2g$ conditions that uniquely determines the harmonic component of the current $\boldsymbol{j}_H$ suffices (see Theorems 6.5 and 7.4).

5 The Perfect Conductor

We turn now to the problem of scattering from a perfect conductor, which requires an analysis of the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$. These are easily expressed in terms of potentials using Lemma 3.1.

THEOREM 5.1 *Let* $\boldsymbol{x}_0 \in \Gamma$. *The limiting values of the tangential components of* $\boldsymbol{E}$ *and* $\boldsymbol{H}$, *in terms of potentials and antipotentials, are given by*

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^\pm} \begin{pmatrix} \boldsymbol{n}\times\boldsymbol{E} \\ \boldsymbol{n}\times\boldsymbol{H} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \pm\boldsymbol{m} \\ \mp\boldsymbol{j} \end{pmatrix} + \begin{pmatrix} -K_1 & 0 & ikK_{2,t} & -K_4 \\ 0 & -K_1 & K_4 & ikK_{2,t} \end{pmatrix} \begin{pmatrix} r \\ q \\ \boldsymbol{j} \\ \boldsymbol{m} \end{pmatrix}. \tag{5.1}$$

where K_1, $K_{2,t}$, *and* K_4 *are defined in Lemma* 3.1.

DEFINITION 5.2 In what follows we denote the operator on the right-hand side of (5.1) as $\mathcal{T}^{\pm}(k)$. We use $\mathcal{T}^{\pm}_{\boldsymbol{E}}(k)$ to denote the first row, and $\mathcal{T}^{\pm}_{\boldsymbol{H}}(k)$ to denote the second.

For scattering from a perfect conductor, let us recall that both (0.8) and (0.9) must hold on Γ. As noted in the introduction, the EFIE approach involves imposing (0.8) using only the classical vector and scalar potentials, $\boldsymbol{A}$ and ϕ, with the physical current $\boldsymbol{J}$ as the unknown. This leads to an integral equation on Γ with a hypersingular kernel that has interior resonances and suffers from low-frequency breakdown. To avoid these difficulties, we turn again to the Debye sources. A nonstandard feature of our approach is that we extract only one scalar equation from the tangential conditions on the $\boldsymbol{E}$ field and couple it to the normal condition (0.9) satisfied by $\boldsymbol{H}$.

5.1 The Hybrid System

The operator defining the tangential components of $\boldsymbol{E}_\pm$ is given by

$$\mathcal{T}_{\boldsymbol{E}}^{\pm}(k)\begin{pmatrix} r \\ q \\ \boldsymbol{j} \\ \boldsymbol{m} \end{pmatrix} = \pm\frac{1}{2}\boldsymbol{m} + \begin{pmatrix} -K_1 & 0 & ikK_{2,t} & -K_4 \end{pmatrix}\begin{pmatrix} r \\ q \\ \boldsymbol{j} \\ \boldsymbol{m} \end{pmatrix}. \tag{5.2}$$

If we restrict to $\boldsymbol{j} = \boldsymbol{j}_{\boldsymbol{R}}(r, q, k)$ and $\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{j}$, then, acting on $(r, q) \in \mathcal{M}_{\Gamma,0}$, the only term of nonnegative order is $-K_1 r$. We use $\mathcal{T}_{\boldsymbol{E}}^{\pm}(k)(r, q)$ to denote this operator restricted to this subspace of data. In order to recast K_1 in (5.2) as a Fredholm operator of the second kind, it is convenient to multiply it by a left parametrix, based on the following standard result:

LEMMA 5.3 *Let $\Gamma \subset \mathbb{R}^3$ be a smooth, connected, closed surface, let $\boldsymbol{x}_0 \in \Gamma$, and let G_0 denote the single-layer potential operator based on the Green's function for the Laplace equation*:

$$G_0[f](\boldsymbol{x}_0) = \int_\Gamma g_0(\boldsymbol{x} - \boldsymbol{x}_0) f(\boldsymbol{x})\, dA(\boldsymbol{x}), \tag{5.3}$$

and let ϕ denote the usual scalar potential with density r. Then

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0^{\pm}} G_0[\Delta_\Gamma \phi](\boldsymbol{x}) = \frac{1}{4} r(\boldsymbol{x}_0) + \widetilde{K}_1[r](\boldsymbol{x}_0) \tag{5.4}$$

where $\widetilde{K}_1$ is an operator of negative order.

PROOF: To see this, we observe that the surface Laplacian satisfies the identity

$$\Delta_\Gamma \phi = \Delta\phi - 2H\frac{\partial \phi}{\partial n} - \frac{\partial^2 \phi}{\partial n^2},$$

where H denotes mean curvature (see, for example, [23]). Since $\Delta\phi = -k^2\phi$ (by construction), we have

$$\Delta_\Gamma \phi = -k^2\phi - 2H\frac{\partial \phi}{\partial n} - \frac{\partial^2 \phi}{\partial n^2}. \tag{5.5}$$

The composition of G_0 with the first two terms on the right-hand side of (5.5) are of order -2 and -1, respectively, hence compact. It is a classical result (a *Calderon relation*) that the composition of a single-layer potential with the second normal derivative of ϕ yields a compact perturbation of $\frac{1}{4}I$ [23]:

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0^{\pm}} G_0\left[\frac{\partial^2 \phi}{\partial n^2}\right](\boldsymbol{x}) = -\frac{1}{4} r(\boldsymbol{x}_0) + K_c[r](\boldsymbol{x}_0),$$

where K_c is an operator of order -1, which completes the proof of the lemma. $\square$

LEMMA 5.4 *Let $\Gamma \subset \mathbb{R}^3$ be a smooth, connected closed surface, let $\boldsymbol{x}_0 \in \Gamma$, and let G_0 denote the single-layer potential operator* (5.3)*. Then*

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm}} G_0 \nabla_\Gamma \cdot [\boldsymbol{n} \times \mathcal{T}_{\boldsymbol{E}}^{\pm}(k)] \begin{pmatrix} r \\ q \end{pmatrix} = -\frac{1}{4} r(\boldsymbol{x}_0) + N_1(k) \begin{pmatrix} r \\ q \end{pmatrix}, \tag{5.6}$$

where $N_1(k)$ is an analytic family of operators of order -1.

PROOF: The result follows from Definition 5.2, the fact that

$$\nabla_\Gamma \cdot \boldsymbol{n} \times K_1[r] = -\Delta_\Gamma \phi[r],$$

and the preceding lemma. □

Recalling that

$$\boldsymbol{j}_R(r,q,0) = 0,$$

we see that

$$G_0 \nabla_\Gamma \cdot [\boldsymbol{n} \times \mathcal{T}_{\boldsymbol{E}}^{\pm}](0) \begin{pmatrix} r \\ q \end{pmatrix} = G_0[\Delta_\Gamma G_0\, r]. \tag{5.7}$$

If Γ has M components, then the null space of this operator is M-dimensional. It is generated by functions r such that $G_0 r$ is constant on each component of Γ. As a consequence of theorem 5.7 in [8], it follows that this null space only intersects $\mathcal{M}_{\Gamma,0}$ at 0.

DEFINITION 5.5 Taking the integral operator in Lemma 5.4 and the integral operator $\mathcal{N}_{\boldsymbol{H}}$ from Definition 4.2, we define $\mathcal{Q}^{\pm}(k)$ as the following hybrid system of integral operators:

$$\mathcal{Q}^{\pm}(k) \begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} G_0 \nabla_\Gamma \cdot [\boldsymbol{n} \times \mathcal{T}_{\boldsymbol{E}}^{\pm}](k) \\ \mathcal{N}_{\boldsymbol{H}}^{\pm}(k) \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix}. \tag{5.8}$$

As divergences, the range of $\nabla_\Gamma \cdot [\boldsymbol{n} \times \mathcal{T}_{\boldsymbol{E}}^{\pm}](k)$ consists of functions of mean zero.

PROPOSITION 5.6 *The family of operators $\mathcal{Q}^{\pm}(k)$ is analytic in k and Fredholm of the second kind. There is a discrete subset $F_+ \subset \mathbb{C}$ such that $\mathcal{Q}^{+}(k) : \mathcal{M}_{\Gamma,0} \to \mathcal{M}_{\Gamma,1}$ is invertible for $k \notin F_+$, where $\mathcal{M}_{\Gamma,1}$ is the L^2-closure of*

$$\{(G_0 r, q) : (r,q) \in \mathcal{M}_{\Gamma,0}\}. \tag{5.9}$$

PROOF: The analyticity statement is immediate from the formula. Examining $\mathcal{Q}^{\pm}(k)$, we see that

$$\mathcal{Q}^{\pm}(k) \begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} \frac{-1}{4} & 0 \\ 0 & \frac{\mp 1}{2} \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix} + \widetilde{N}_1^{\pm}(k) \begin{pmatrix} r \\ q \end{pmatrix}, \tag{5.10}$$

where $\widetilde{N}_1^{\pm}$ is an analytic family of operators of order -1. Thus, $\mathcal{Q}^{\pm}(k)$ is Fredholm of the second kind. The last statement follows from the facts that $\mathcal{Q}^{+}(0) : \mathcal{M}_{\Gamma,0} \to \mathcal{M}_{\Gamma,1}$ is invertible and $\mathcal{Q}^{+}(k)\mathcal{M}_{\Gamma,0} \subset \mathcal{M}_{\Gamma,1}$. □

We may now state our principal result in the simply connected case.

THEOREM 5.7 *If Γ is simply connected, then F_+ is disjoint from the closed upper half-plane. Thus, the integral equation*

$$\mathcal{Q}^+(k)\begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} \tag{5.11}$$

provides a unique solution to the scattering problem from a perfect conductor for any k in the closed upper half-plane. Here

$$f = G_0 \nabla_\Gamma \cdot [\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{E}^{\mathrm{in}}], \quad h = \boldsymbol{n} \cdot \boldsymbol{H}^{\mathrm{in}} \restriction_{T\Gamma}. \tag{5.12}$$

This is proved as Theorem 7.7 in Section 7.1. We leave the discussion of applying our method in the non-simply-connected case to Section 7.1.

5.2 Low-Frequency Behavior in the Simply Connected Case

The representation of solutions to the THME(k), using data from $\mathcal{M}_{\Gamma,0} \oplus \mathcal{H}^1(\Gamma)$, afforded by (6.30) behaves well as the frequency tends to 0. In the simply connected case we only have data from $\mathcal{M}_{\Gamma,0}$. As k tends to 0, this space of solutions tends to the orthogonal complement of the harmonic Dirichlet fields, that is, outgoing harmonic fields with vanishing tangential components on $b\Omega$, so that $\boldsymbol{E}$ and $\boldsymbol{H}$ are recovered from $-\nabla\phi$ and $-\nabla\phi_M$ alone. This is proven, along with the multiply connected case, in Section 7.2. It is therefore apparent that $\mathcal{Q}^+(k)$ provides a means for finding and representing solutions to the perfect conductor problem, which has neither interior resonances nor suffers from low-frequency breakdown.

6 Exterior Form Representation

For the remainder of this paper we represent the electric field $\boldsymbol{E}$ as a 1-form $\boldsymbol{\xi}$ and the magnetic field $\boldsymbol{H}$ as a 2-form $\boldsymbol{\eta}$. If

$$\boldsymbol{E} = e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3} \quad \text{and} \quad \boldsymbol{H} = h_1\partial_{x_1} + h_2\partial_{x_2} + h_3\partial_{x_3}, \tag{6.1}$$

then

$$\begin{aligned} \boldsymbol{\xi} &= e_1 dx_1 + e_2 dx_2 + e_3 dx_3, \\ \boldsymbol{\eta} &= h_1 dx_2 \wedge dx_3 + h_2 dx_3 \wedge dx_1 + h_3 dx_1 \wedge dx_2. \end{aligned} \tag{6.2}$$

If $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product, i.e., the metric on $\mathbb{R}^3$, then $\boldsymbol{\xi}$ is defined by the condition that, for every vector field $\boldsymbol{V}$, we have

$$\langle \boldsymbol{V}, \boldsymbol{E} \rangle = \boldsymbol{\xi}(\boldsymbol{V}). \tag{6.3}$$

That is, $\boldsymbol{\xi}$ is the metric dual of $\boldsymbol{E}$, and $\star\boldsymbol{\eta}$ (with $\star$ the Hodge star operator, see Remark 6.1 and [31]) is the metric dual of $\boldsymbol{H}$.

Remark 6.1. In much of the paper we need to distinguish between the Hodge star operator acting on forms defined on $\mathbb{R}^3$ and that acting on forms defined on surfaces in $\mathbb{R}^3$. We denote the $\mathbb{R}^3$ operator by $\star_3$ and a surface operator by $\star_2$. The intended surface should be clear from the context.

If Γ is a smooth closed surface in $\mathbb{R}^3$, which bounds a region D, then it obtains an orientation from its embedding into $\mathbb{R}^3$. Let $\boldsymbol{n}$ be the outward-pointing unit normal vector and X_1, X_2 a local oriented orthonormal frame for $T\Gamma$. We let ω_1, ω_2, ν be the local coframe for $T^*\mathbb{R}^3 \restriction_\Gamma$, dual to $X_1, X_2, \boldsymbol{n}$. Note in particular that $\omega_1(\boldsymbol{n}) = \omega_2(\boldsymbol{n}) = 0$. We say that the frame $(X_1, X_2, \boldsymbol{n})$ (or coframe $(\omega_1, \omega_2, \nu)$) is *adapted to* Γ. The 1-form that is the metric dual of the vector field $aX_1 + bX_2$ is $a\omega_1 + b\omega_2$.

The volume form on $\mathbb{R}^3$ and area form on Γ are given locally by

$$dV = \omega_1 \wedge \omega_2 \wedge \nu, \quad dA = i_{\boldsymbol{n}} dV = \omega_1 \wedge \omega_2. \tag{6.4}$$

In terms of the adapted frame, the Hodge star operator on $\mathbb{R}^3$ is

$$\begin{gathered} \star_3 1 = \omega_1 \wedge \omega_2 \wedge \nu, \quad \star_3 \omega_1 \wedge \omega_2 \wedge \nu = 1, \\ \star_3 \omega_1 = \omega_2 \wedge \nu, \quad \star_3 \omega_2 = -\omega_1 \wedge \nu, \quad \star_3 \nu = \omega_1 \wedge \omega_2, \\ \star_3 \omega_1 \wedge \nu = -\omega_2, \quad \star_3 \omega_2 \wedge \nu = \omega_1, \quad \star_3 \omega_1 \wedge \omega_2 = \nu. \end{gathered} \tag{6.5}$$

The Hodge star operator on the surface (oriented as the boundary of D) is given by

$$\begin{gathered} \star_2 1 = \omega_1 \wedge \omega_2, \quad \star_2 \omega_1 \wedge \omega_2 = 1, \\ \star_2 \omega_1 = \omega_2, \quad \star_2 \omega_2 = -\omega_1. \end{gathered} \tag{6.6}$$

It is useful to note that if $\boldsymbol{q}$ is a 1-form defined on Γ, then $\star_2^2 \boldsymbol{q} = -\boldsymbol{q}$. If $\boldsymbol{v}$ is a vector field tangent to Γ and ω, its metric dual, then $\star_2 \omega$ is the metric dual of $\boldsymbol{n} \times \boldsymbol{v}$.

To emphasize the distinction between an exterior form acting on $T\mathbb{R}^3$ restricted to a surface $\Gamma \subset \mathbb{R}^3$ and the restriction of this form to directions tangent to Γ, we sometimes use $\boldsymbol{\alpha} \restriction_\Gamma$ to denote the former notion of restriction, and $\boldsymbol{\alpha} \restriction_{T\Gamma}$ the latter. For a 1-form $\boldsymbol{\alpha}$ represented along Γ in the adapted coframe by $\boldsymbol{\alpha} \restriction_\Gamma = a\omega_1 + b\omega_2 + c\nu$, we have

$$\boldsymbol{\alpha} \restriction_{T\Gamma} \leftrightarrow a\omega_1 + b\omega_2. \tag{6.7}$$

We denote this latter restriction by $\boldsymbol{\alpha}_t$. We use the notation d_Γ to denote the exterior differential acting on forms on Γ.

For a 2-form $\boldsymbol{\beta} \restriction_\Gamma = a\omega_1 \wedge \nu + b\omega_2 \wedge \nu + c\omega_1 \wedge \omega_2$, we have

$$\boldsymbol{\beta} \restriction_{T\Gamma} \leftrightarrow c\omega_1 \wedge \omega_2. \tag{6.8}$$

For a 3-form $\boldsymbol{\gamma}$, dimensional considerations imply that

$$\boldsymbol{\gamma} \restriction_{T\Gamma} \equiv 0. \tag{6.9}$$

If f is a 0-form, or scalar function, then

$$f \restriction_{T\Gamma} = f \restriction_\Gamma. \tag{6.10}$$

The curl part of the time-harmonic Maxwell's equations takes the form

$$d\boldsymbol{\xi} = ik\boldsymbol{\eta}, \quad d^*\boldsymbol{\eta} = -ik\boldsymbol{\xi}. \tag{6.11}$$

For $k \neq 0$, these equations imply the divergence equations, which take the form

$$d^*\boldsymbol{\xi} = 0, \quad d\boldsymbol{\eta} = 0. \tag{6.12}$$

An outgoing solution to the Helmholtz equation on 1-forms satisfies the analogue of the Silver-Müller radiation conditions:

$$i_{\hat{\boldsymbol{x}}}\, d\boldsymbol{\xi} - d^*\boldsymbol{\xi}\hat{\boldsymbol{x}} \cdot d\boldsymbol{x} - ik\boldsymbol{\xi} = o\left(\frac{1}{|x|}\right), \tag{6.13}$$

where $\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$. A magnetic field $\boldsymbol{\eta}$ is outgoing if $\star\boldsymbol{\eta}$ satisfies (6.13).

The standard integration-by-parts formula for outgoing solutions to the vector Helmholtz equation is derived by considering the L^2-norm of the quantity in the radiation condition. It is as follows:

LEMMA 6.2 *If $\boldsymbol{\xi}$ is a 1-form defined in Ω satisfying $\Delta\boldsymbol{\xi} + k^2\boldsymbol{\xi} = 0$ and the radiation condition* (6.13)*, with $\Im(k) \geq 0$, then*

$$\begin{aligned}\lim_{R\to\infty}\Bigg(2\Im(k)\int\limits_{\Omega_R} &\left[\|d\boldsymbol{\xi}\|^2 + \|d^*\boldsymbol{\xi}\|^2 + |k|^2\|\boldsymbol{\xi}\|^2\right] dV \\ &+ \int\limits_{S_R}\left[\|i_{\hat{\boldsymbol{x}}} d\boldsymbol{\xi} - d^*\boldsymbol{\xi}\,\hat{\boldsymbol{x}} \cdot d\boldsymbol{x}\|^2 + |k|^2\|\boldsymbol{\xi}\|^2\right] dA\Bigg) \\ &= -2\Im\Bigg[k\int\limits_{\Gamma}\left[\langle\boldsymbol{\xi}, i_{\boldsymbol{n}} d\bar{\boldsymbol{\xi}}\rangle - \langle i_{\boldsymbol{n}}\boldsymbol{\xi}, d^*\bar{\boldsymbol{\xi}}\rangle\right] dA\Bigg].\end{aligned} \tag{6.14}$$

Remark 6.3. If $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is a solution to the THME(k) in Ω, then the outgoing radiation condition can be rewritten, in agreement with (0.6), as

$$i_{\hat{\boldsymbol{x}}}\boldsymbol{\eta} - \boldsymbol{\xi} = o\left(\frac{1}{|x|}\right). \tag{6.15}$$

6.1 Uniqueness for Maxwell's Equations

If $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is a solution to the Maxwell system, then $d^*\boldsymbol{\xi} = 0$ and the boundary term in (6.14) reduces to

$$-2\Im\left(k\int\limits_{\Gamma}\langle\boldsymbol{\xi}, i_{\boldsymbol{n}} d\bar{\boldsymbol{\xi}}\rangle dA\right). \tag{6.16}$$

Let ν denote a 1-form defined along Γ, which restricts to 0 on $T\Gamma$ and is normalized by $\nu(\boldsymbol{n}) = 1$.

The definition of the inner product on forms, the fact that $\langle\nu, i_{\boldsymbol{n}} d\bar{\boldsymbol{\xi}}\rangle = 0$, and the definition of $\star_2$ on Γ imply the identity

$$-2\Im\left(k\int\limits_{\Gamma}\langle\boldsymbol{\xi}, i_{\boldsymbol{n}} d\bar{\boldsymbol{\xi}}\rangle dA\right) = -2\Im\left(k\int\limits_{\Gamma}[\boldsymbol{\xi}\restriction_{\Gamma}] \wedge \star_2[i_{\boldsymbol{n}} d\bar{\boldsymbol{\xi}}\restriction_{\Gamma}]\right). \tag{6.17}$$

Using this identity and Lemma 6.2 we can prove the basic uniqueness theorems. Note that $\boldsymbol{\xi}$ is a 1-form so $\boldsymbol{\xi}_t = \boldsymbol{\xi} \restriction_{T\Gamma}$ corresponds to the tangential components of $\boldsymbol{E}$ and $i_{\boldsymbol{n}}\boldsymbol{\xi} \restriction_{\Gamma}$, the normal component. The magnetic field $\boldsymbol{\eta}$ is a 2-form and therefore $\boldsymbol{\eta} \restriction_{T\Gamma}$ gives the normal component, and $[i_{\boldsymbol{n}}\boldsymbol{\eta}]_t = i_{\boldsymbol{n}}\boldsymbol{\eta} \restriction_{T\Gamma}$ gives the data in the tangential components corresponding to $n \times \boldsymbol{H}$.

We restate the classical result (Theorem 2.2) that an outgoing solution to the THME(k) is determined by either the tangential components of the electric or magnetic fields in the language of forms:

THEOREM 6.4 *Suppose that* $(\boldsymbol{\xi}, \boldsymbol{\eta})$ *is an outgoing solution to the THME*(k), *in* Ω, *for a* $k \neq 0$, *in* $\overline{\mathbb{C}_+}$. *If either* $\boldsymbol{\xi}_t$ *or* $[i_{\boldsymbol{n}}\boldsymbol{\eta}]_t$ *vanish, then the solution is identically zero in* Ω.

Kress' result (Theorem 2.5) on the normal components of $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is restated as follows:

THEOREM 6.5 *Suppose that* $(\boldsymbol{\xi}, \boldsymbol{\eta})$ *is an outgoing solution to the THME*(k) *in* Ω *for* $k \neq 0$ *in* $\overline{\mathbb{C}_+}$. *If every component of the boundary of* Ω *is simply connected, then the solution is determined by the normal components* $i_{\boldsymbol{n}}\boldsymbol{\xi} \restriction_{\Gamma}$ *and* $\boldsymbol{\eta} \restriction_{T\Gamma}$. *If the sum of the genera of the components of* Γ *equals* $g > 0$, *then there is a subspace of outgoing solutions to THME*(k) *with*

$$i_{\boldsymbol{n}}\boldsymbol{\xi} \restriction_{\Gamma} = 0 \quad \text{and} \quad \boldsymbol{\eta} \restriction_{T\Gamma} = 0 \tag{6.18}$$

of dimension $2g$.

Remark 6.6. As noted above, we call solutions to THME(k) that satisfy (6.18) *k-Neumann fields*, and denote the space of such solutions by $\mathcal{H}_k(\Omega)$. Here we give a bound on $\dim \mathcal{H}_k(\Omega)$ and a novel description of the additional data needed to specify the projection into this space. Later in the paper we give a new proof that $\dim \mathcal{H}_k(\Omega) = 2g$ for k in the closed upper half-plane $\overline{\mathbb{C}_+}$. In this regard, the case $k = 0$ is classical. As noted above, this result was proved in [17].

PROOF: Suppose that $i_{\boldsymbol{n}}\boldsymbol{\xi}$ and $\boldsymbol{\eta} \restriction_{\Gamma}$ both vanish. Let $\boldsymbol{\alpha} = \boldsymbol{\xi} \restriction_{\Gamma}$ and $\boldsymbol{\beta} = \star_3 \boldsymbol{\eta} \restriction_{\Gamma}$. The usual properties of the exterior derivative, the hypothesis $\boldsymbol{\eta} \restriction_{T\Gamma} = 0$, and the equation $d\boldsymbol{\xi} = ik\boldsymbol{\eta}$ imply that

$$d_\Gamma \boldsymbol{\alpha} = 0. \tag{6.19}$$

We can rewrite $d^*\boldsymbol{\eta} = -ik\boldsymbol{\xi}$ as $d \star_3 \boldsymbol{\eta} = -ik \star_3 \boldsymbol{\xi}$. The hypothesis $i_{\boldsymbol{n}}\boldsymbol{\xi} = 0$ now implies that

$$d_\Gamma \boldsymbol{\beta} = 0. \tag{6.20}$$

A calculation using a coframe adapted to Γ shows that

$$\star_2 \boldsymbol{\beta} = -[i_{\boldsymbol{n}}\boldsymbol{\eta}] \restriction_{\Gamma}; \tag{6.21}$$

see (6.44). The equation $d\boldsymbol{\xi} = ik\boldsymbol{\eta}$ implies that

$$[i_{\boldsymbol{n}} d\boldsymbol{\xi}] \restriction_{\Gamma} = -ik \star_2 \boldsymbol{\beta}. \tag{6.22}$$

We can therefore express the right-hand side of (6.17) as

$$2|k|^2\Re\left(\int_\Gamma \boldsymbol{\alpha}\wedge\bar{\boldsymbol{\beta}}\right). \tag{6.23}$$

If Γ is simply connected, then the equation $d_\Gamma\boldsymbol{\alpha} = 0$ implies that $\boldsymbol{\alpha} = d_\Gamma u$. As $d_\Gamma\bar{\boldsymbol{\beta}} = 0$ as well, a simple application of Stokes formula shows that

$$\int_\Gamma d_\Gamma u\wedge\bar{\boldsymbol{\beta}} = 0. \tag{6.24}$$

This completes the proof of the theorem when Γ is simply connected.

For the general case, let $H^1_{\mathrm{dR}}(\Gamma)$ denote the de Rham cohomology group with $\dim H^1_{\mathrm{dR}}(\Gamma) = 2g \neq 0$. We show that the space of solutions with vanishing normal components, for which the integral in (6.23) is nonvanishing, depends on at most $2g$ parameters. Using the wedge product, we define a pairing W on closed 1-forms:

$$W(\eta,\omega) = \int_\Gamma \eta\wedge\omega. \tag{6.25}$$

If $d_\Gamma\eta = 0$ and $\omega = d_\Gamma u$, then, as noted above, Stokes' theorem implies that

$$W(\eta,\omega) = 0. \tag{6.26}$$

Hence W defines a skew-symmetric form on $H^1_{\mathrm{dR}}(\Gamma)$, which is well-known to be nondegenerate. As the $\dim H^1_{\mathrm{dR}}(\Gamma) = 2g$, this observation completes the proof of the fact that $\dim\mathcal{H}_k(\Omega) \le 2g$. If the image of either $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ in $H^1_{\mathrm{dR}}(\Gamma)$ vanishes, then $W(\boldsymbol{\alpha},\bar{\boldsymbol{\beta}}) = 0$, which implies, as above, that the solution in Ω is 0. □

From this proof we see that the image of either $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ in $H^1(\Gamma)$ provides data specifying a k-Neumann field. This should be contrasted with the data used by Kress, given in equation (2.2). As $\dim\mathcal{H}_k(\Omega) = \dim H^1(\Gamma)$, the maps from $\mathcal{H}_k(\Omega)$ to $H^1(\Gamma)$ defined by $(\boldsymbol{\xi},\boldsymbol{\eta})\mapsto\boldsymbol{\alpha}$ and $(\boldsymbol{\xi},\boldsymbol{\eta})\mapsto\boldsymbol{\beta}$ are both isomorphisms when $k\neq 0$.

We complete this section by proving Lemma 2.6, which shows that the normal components of $(\boldsymbol{\xi},\boldsymbol{\eta})$, a solution to THME($k$) with $k\neq 0$, have vanishing mean value over every component of the boundary. While superficially this might appear analogous to the fact that the normal derivative of a harmonic function in a bounded domain has mean value over the boundary, it is actually an elementary consequence of the equations themselves and Stokes' theorem on a closed surface.

If $k\neq 0$, then the Maxwell equations (6.11) imply that

$$\star_3\boldsymbol{\xi}\restriction_\Gamma = \frac{-1}{ik}d_\Gamma[\star_3\boldsymbol{\eta}\restriction_\Gamma] \quad \boldsymbol{\eta}\restriction_\Gamma = \frac{1}{ik}d_\Gamma[\boldsymbol{\xi}\restriction_\Gamma]. \tag{6.27}$$

As

$$\star_3\boldsymbol{\xi}\restriction_\Gamma = i_{\boldsymbol{n}}\boldsymbol{\xi}\,dA \tag{6.28}$$

the relations in (6.27) imply that these forms are exact and therefore Stokes' theorem implies that a solution of THME(k) with $k \neq 0$ satisfies

$$\int_{\Gamma_m} i_{\boldsymbol{n}} \boldsymbol{\xi} \, dA = \int_{\Gamma_m} \boldsymbol{\eta} = 0 \tag{6.29}$$

for $m = 1, \ldots, M$. Note that this is true whether the limit is taken from Ω or D. This completes the proof of Lemma 2.6, restated as follows:

PROPOSITION 6.7 *Let* $(\boldsymbol{\xi}, \boldsymbol{\eta})$ *be a solution to the THME*(k) *for* $k \neq 0$ *in a region* $G \subset \mathbb{R}^3$ *with a smooth, bounded boundary. The normal components* $(i_{\boldsymbol{n}} \boldsymbol{\xi}, i_n \star_3 \boldsymbol{\eta})$ *have mean* 0 *over every component of* Γ.

6.2 Potentials and Boundary Integral Equations

We now reexpress (0.16) in terms of exterior forms. Assuming, as before, that the time dependence is $e^{-i\omega t}$, the permittivity is ϵ, the permeabilty is μ, and $k = \omega\sqrt{\epsilon\mu}$, we set

$$\boldsymbol{\xi} = (ik\boldsymbol{\alpha} - d\phi - d^*\boldsymbol{\alpha}_m), \quad \boldsymbol{\eta} = (d\boldsymbol{\alpha} + ik\boldsymbol{\alpha}_m + d^*\Phi_m), \tag{6.30}$$

where ϕ is a scalar function, $\boldsymbol{\alpha}$ a 1-form, $\boldsymbol{\alpha}_m$ a 2-form, and $\Phi_m = \phi_m dV$ a 3-form. This representation is quite similar to what one obtains using the fundamental solution for the Dirac operator $d + d^*$; see [3].

In order for $(\boldsymbol{\xi}, \boldsymbol{\eta})$ to satisfy the THME(k), the potentials must satisfy

$$d^*\boldsymbol{\alpha} = -ik\phi, \quad d\boldsymbol{\alpha}_m = ik\Phi_m. \tag{6.31}$$

As before, $g_k(x - y)$ denotes the outgoing fundamental solution for the scalar Helmholtz equation with frequency k. As discussed in the introduction, *all* of the potentials are expressed in terms of a pair of 1-forms $\boldsymbol{j}, \boldsymbol{m}$ defined on Γ, though in the end, we do *not* use $\boldsymbol{j}$ and $\boldsymbol{m}$ as the "fundamental" parameters. When we express these 1-forms in terms of the ambient basis from $\mathbb{R}^3$, e.g.,

$$\boldsymbol{j} = j_1(x)dx_1 + j_2(x)dx_2 + j_3(x)dx_3, \tag{6.32}$$

we normalize with the requirement

$$i_{\boldsymbol{n}} \boldsymbol{j} = \boldsymbol{j}(\boldsymbol{n}) \equiv 0. \tag{6.33}$$

These 1-forms are the metric duals of the vector fields, tangent to Γ, previously denoted by $\boldsymbol{j}$ and $\boldsymbol{m}$.

The "vector" potentials are given in terms of surface integrals by setting

$$\begin{aligned} \boldsymbol{\alpha} &= \int_{\Gamma} g_k(x - y)[j_1(y)dx_1 + j_2(y)dx_2 + j_3(y)dx_3]dA(y), \\ \boldsymbol{\alpha}_m &= \star_3\Bigg[\int_{\Gamma} g_k(x - y)[m_1(y)dx_1 + m_2(y)dx_2 + m_3(y)dx_3]dA(y)\Bigg]. \end{aligned} \tag{6.34}$$

Using the equations in (6.31) we obtain the form of the potentials defining ϕ and $\Phi_m = \phi_m dV$, letting

$$\begin{aligned} \phi(x) &= \int_\Gamma g_k(x-y)r(y)dA(y), \\ \phi_m(x) &= \int_\Gamma g_k(x-y)q(y)dA(y), \end{aligned} \tag{6.35}$$

where we let

$$\frac{1}{ik} d_\Gamma \star_2 \boldsymbol{j} = r\,dA, \quad \frac{1}{ik} d_\Gamma \star_2 \boldsymbol{m} = q\,dA. \tag{6.36}$$

The scalar functions (r,q) are, as before, the Debye sources. From this definition and Stokes' theorem, we see that the mean values of r and q vanish on every connected component Γ_j of Γ,

$$\int_{\Gamma_j} r\,dA = \int_{\Gamma_j} q\,dA = 0. \tag{6.37}$$

This proves Lemma 1.4. For $(\boldsymbol{\xi}, \boldsymbol{\eta})$ to satisfy the Maxwell equations, it is *necessary* for the conditions in (6.36) to hold.

As before, we let $\mathcal{M}_{\Gamma,0}$ denote pairs of functions (r,q) defined on Γ with mean 0 on every component of Γ. If we assume that $\boldsymbol{m} = \star_2 \boldsymbol{j}$ (as we usually do), then, taking into account the fact that the (negative) Laplace operator on 1-forms is given by $-\Delta_1 = d_\Gamma^* d_\Gamma + d_\Gamma d_\Gamma^*$, we obtain the relation

$$\Delta_1 \boldsymbol{j} = ik[d_\Gamma r - \star_2 d_\Gamma q]. \tag{6.38}$$

If the components of Γ are all of genus 0, then equation (6.38) always has a unique solution. If Γ has positive genus components, then one needs to deal with the null space of Δ_1.

If $H^1_{\mathrm{dR}}(\Gamma) \neq 0$, then the null space of Δ_1, $\mathcal{H}^1(\Gamma)$, which agrees with the space of solutions to

$$d_\Gamma \boldsymbol{\alpha} = 0, \quad d_\Gamma^* \boldsymbol{\alpha} = 0, \tag{6.39}$$

is isomorphic to $H^1_{\mathrm{dR}}(\Gamma)$. These are the harmonic 1-forms. The right-hand side in (6.38) is orthogonal to $\mathcal{H}^1(\Gamma)$ and hence lies in the range of Δ_1. Let R_1 denote the partial inverse of the Laplacian on 1-forms, with range orthogonal to $\mathcal{H}^1(\Gamma)$, and set

$$\boldsymbol{j}_R(r,q,k) = ikR_1[d_\Gamma r - \star_2 d_\Gamma q]. \tag{6.40}$$

Because the ranges of d_Γ and d_Γ^* are orthogonal to the null space of Δ_1, this equation is solvable whether or not r and q satisfy the mean value condition. Note that the solution to (6.40) tends to 0 as $k \to 0$.

Using the relations $\Delta_1 d_\Gamma = d_\Gamma \Delta_0$ and $\Delta_2 = \star_2 \Delta_0 \star_2$, we can re-express $\boldsymbol{j}_R$ in the form

$$\boldsymbol{j}_R(r,q,k) = ik[d_\Gamma R_0 r - \star_2 d_\Gamma R_0 q]. \tag{6.41}$$

Here R_0 is the partial inverse of Δ_0, which annihilates functions constant on each component of Γ and has range equal to the set of functions with mean zero on each component of Γ. Equation (6.41) shows that this approach to representing solutions of Maxwell's equations in terms of the pair (r,q) only requires an inverse for the *scalar* Laplacian on Γ.

For any (r,q), the solution space to (6.38) is isomorphic to $\mathcal{H}^1(\Gamma)$. Adding a harmonic 1-form to $\boldsymbol{j}_R$ does not change r and q, though it changes the fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ and plays a central role in the discussion of $\mathcal{H}_k(\Omega)$. If $g \neq 0$, then the space of outgoing solutions to THME(k) is parametrized by $\mathcal{M}_{\Gamma,0} \oplus \mathcal{H}^1(\Gamma)$. So given data $(r,q,\boldsymbol{j}_H)$, we often speak of the solution to the THME(k) "defined" by this data. If $\boldsymbol{j}_H$ is missing, then it should be understood to be 0; i.e., the solution "defined by (r,q)" is the solution defined by $(r,q,0)$ in the sense above.

6.3 Boundary Equations and Jump Relations

As with the vector field representation, we can take the limits of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in (6.30) as the point of evaluation approaches Γ and obtain boundary integral equations for the normal and tangential components of these forms. Indeed, there is no necessity to rewrite these equations, we simply use

$$\mathcal{T}^{\pm}_{\boldsymbol{\xi}}(k)(r,q,\boldsymbol{j},\boldsymbol{m}), \quad \mathcal{T}^{\pm}_{\boldsymbol{\eta}}(k)(r,q,\boldsymbol{j},\boldsymbol{m}),$$

to denote the limiting tangential components, and the limiting normal components are $\mathcal{N}^{\pm}_{\boldsymbol{\xi}}(k)(r,q,\boldsymbol{j},\boldsymbol{m})$ and $\mathcal{N}^{\pm}_{\boldsymbol{\eta}}(k)(r,q,\boldsymbol{j},\boldsymbol{m})$. Keep in mind that, in the vector field representation, the tangential components are represented as the limits of $\boldsymbol{n} \times \boldsymbol{E}$ and $\boldsymbol{n} \times \boldsymbol{H}$, which correspond to $\star_2 \boldsymbol{\xi}_t$ and $\star_2([\star_3 \boldsymbol{\eta}]_t)$, respectively. For consistency, we use $\mathcal{T}^{\pm}_{\boldsymbol{\xi}}$ and $\mathcal{T}^{\pm}_{\boldsymbol{\eta}}$ to denote the boundary values of these quantities. As before, $+$ indicates the limit taken from Ω and $-$ the limit taken from D. If $\boldsymbol{j} = \boldsymbol{j}_R(r,q,k)$ and $\boldsymbol{m} = \star_2 \boldsymbol{j}$, then we omit them from the argument list; e.g., to denote $\mathcal{T}^{\pm}_{\boldsymbol{\xi}}(k)(r,q,\boldsymbol{j}_R(r,q,k),\star_2 \boldsymbol{j}_R(r,q,k))$, etc., we use the abbreviated notation $\mathcal{T}^{\pm}_{\boldsymbol{\xi}}(k)(r,q)$.

Below we use the jump relations to prove a uniqueness theorem. So it is useful to reformulate them in the form language. The magnetic field is represented by the 2-form $\boldsymbol{\eta} = h_1\, dx_2 \wedge dx_3 + h_2\, dx_3 \wedge dx_1 + h_3\, dx_1 \wedge dx_2$, so that $\star_3 \boldsymbol{\eta} = h_1\, dx_1 + h_2\, dx_2 + h_3\, dx_3$. The most direct way to define the normal and tangential components of $\boldsymbol{\eta}$ along Γ is as $i_{\boldsymbol{n}} \star_3 \boldsymbol{\eta}_\pm$ and $(\star_3 \boldsymbol{\eta}_\pm)_t$. In this formulation the jump relations then take the form

$$\begin{gathered} i_{\boldsymbol{n}}(\boldsymbol{\xi}_+ - \boldsymbol{\xi}_-) = r, \quad (\boldsymbol{\xi}_+ - \boldsymbol{\xi}_-)_t = \star_2 \boldsymbol{m}, \\ i_{\boldsymbol{n}}(\star_3 \boldsymbol{\eta}_+ - \star_3 \boldsymbol{\eta}_-) = q, \quad (\star_3 \boldsymbol{\eta}_+ - \star_3 \boldsymbol{\eta}_-)_t = -\star_2 \boldsymbol{j}. \end{gathered} \tag{6.42}$$

It is also useful to calculate the relationship between $i_{\boldsymbol{n}}\boldsymbol{\eta}$ and $(\star_3\boldsymbol{\eta})_t$. In an adapted frame $(\omega_1, \omega_2, \nu)$, we have

$$\boldsymbol{\eta} = a\omega_1 \wedge \omega_2 + b\nu \wedge \omega_1 + c\nu \wedge \omega_2 \tag{6.43}$$

and therefore

$$\begin{cases} i_{\boldsymbol{n}}\boldsymbol{\eta} = b\omega_1 + c\omega_2, \\ (\star_3\boldsymbol{\eta})_t = b\omega_2 - c\omega_1, \end{cases} \qquad \text{which implies that } \star_2 [i_{\boldsymbol{n}}\boldsymbol{\eta}] = (\star_3\boldsymbol{\eta})_t. \tag{6.44}$$

7 Uniqueness for the Tangential Equations

Suppose that there is a $k \in \overline{\mathbb{C}_+} \setminus \{0\}$ and nontrivial data $(r, q) \in \mathcal{M}_{\Gamma,0}$ and $\boldsymbol{j}$ satisfying (6.36), with $\boldsymbol{m} = \star_2\boldsymbol{j}$, so that

$$\mathcal{T}_{\boldsymbol{\xi}}^{+}(k)(r, q, \boldsymbol{j}, \boldsymbol{m}) = 0. \tag{7.1}$$

Let $(\boldsymbol{\xi}_\pm, \boldsymbol{\eta}_\pm)$ be the solutions to the Maxwell equations defined by this data in the complement of Γ. By their definition it is clear that the tangential components of $\boldsymbol{\xi}_+$ vanish along Γ. As $\Im(k) \geq 0$, the solution in Ω is outgoing and Theorem 6.4 shows that $(\boldsymbol{\xi}_+, \boldsymbol{\eta}_+) \equiv (0, 0)$. The jump relations, (6.42), and the fact that $\boldsymbol{m} = \star_2\boldsymbol{j}$ allow us to determine the tangential boundary data for $(\boldsymbol{\xi}_-, \boldsymbol{\eta}_-)$:

$$\boldsymbol{\xi}_- \restriction_{T\Gamma} = \boldsymbol{j}, \quad i_{\boldsymbol{n}}\boldsymbol{\eta}_- \restriction_{T\Gamma} = \boldsymbol{j}. \tag{7.2}$$

It is not difficult to see that the boundary condition on the Maxwell system in D implied by these relations,

$$\boldsymbol{\xi}_- \restriction_{T\Gamma} = i_{\boldsymbol{n}}\boldsymbol{\eta}_- \restriction_{T\Gamma}, \tag{7.3}$$

is not formally self-adjoint!

Observe that $d^*d\boldsymbol{\xi}_- = k^2\boldsymbol{\xi}_-$ and $i_{\boldsymbol{n}}d\boldsymbol{\xi}_- \restriction_\Gamma = iki_{\boldsymbol{n}}\boldsymbol{\eta}_- \restriction_\Gamma$. Using a standard integration-by-parts formula, we obtain

$$\begin{aligned} \int_D (d\boldsymbol{\xi}_-, d\boldsymbol{\xi}_-)dV &= \int_D (d^*d\boldsymbol{\xi}_-, \boldsymbol{\xi}_-)dV + \int_{bD} (i_{\boldsymbol{n}}d\boldsymbol{\xi}_-, \boldsymbol{\xi}_-)dA \\ &= k^2 \int_D (\boldsymbol{\xi}_-, \boldsymbol{\xi}_-)dV + ik \int_{bD} (i_{\boldsymbol{n}}\boldsymbol{\eta}_- \restriction_\Gamma, \boldsymbol{\xi}_- \restriction_\Gamma)dA. \end{aligned} \tag{7.4}$$

Combining this with (7.2) gives

$$-ik \int_{bD} (\boldsymbol{j}, \boldsymbol{j})dA = k^2 \int_D (\boldsymbol{\xi}_-, \boldsymbol{\xi}_-)dV - \int_D (d\boldsymbol{\xi}_-, d\boldsymbol{\xi}_-)dV. \tag{7.5}$$

We can rewrite this relation as

$$-aik = bk^2 - c, \tag{7.6}$$

where a, b, and c are nonnegative real numbers. If b or c vanishes, then it is clear that $\boldsymbol{\xi}_- \equiv 0$. If $a = 0$, then $\boldsymbol{j} \equiv 0$. For a countable set of real numbers $\{k_j\}$, there exist nontrivial solutions to the equations

$$d^* d\boldsymbol{\xi}_- = k_j^2 \boldsymbol{\xi}_-, \quad d^*\boldsymbol{\xi}_- = 0, \quad \boldsymbol{\xi}_- \restriction_{TbD} = 0. \tag{7.7}$$

In the present circumstance, however, (r, q) are generalized Debye sources and therefore

$$r\, dA = \frac{1}{ik} d_\Gamma \star_2 \boldsymbol{j} \quad \text{and} \quad q\, dA = \frac{1}{ik} d_\Gamma \boldsymbol{j}\,. \tag{7.8}$$

If $a = 0$, then all the boundary potentials vanish, and therefore $\boldsymbol{\xi}_- \equiv 0$ as well.

Using the quadratic formula, we see that

$$k_\pm = \frac{-ia \pm \sqrt{4bc - a^2}}{2b}. \tag{7.9}$$

As a, b, and c are all positive, (7.9) shows that $\Im k_\pm < 0$. This argument applies, *mutatis mutandis*, to $\mathcal{T}_{\boldsymbol{\eta}}^+(k)$. Formula (7.9) and the discussion above complete the proof of the following theorem:

THEOREM 7.1 *Assuming that* $\boldsymbol{m} = \star_2 \boldsymbol{j}$ *and* (r, q) *satisfy* (6.36)*, then, for* $\Im k \geq 0$*,* $k \neq 0$*, the null spaces of both* $\mathcal{T}_{\boldsymbol{\xi}}^+(k)$ *are* $\mathcal{T}_{\boldsymbol{\eta}}^+(k)$ *are trivial.*

In the case that Γ is simply connected, this implies that E_+, the exceptional set for $\mathcal{N}^+(k)$, is disjoint from the closed upper half-plane.

COROLLARY 7.2 *If every component of* Γ *is simply connected, then, for* k *with* $\Im k \geq 0$*, the Fredholm operator* $\mathcal{N}^+(k)$ *is an isomorphism from* $\mathcal{M}_{\Gamma,0}$ *to itself. For such* k*, the rows of* $\mathcal{T}^+(k)$ *are also surjective and hence isomorphisms.*

PROOF: If Γ is simply connected, then any 1-form $\boldsymbol{j}$ on Γ has a unique representation as $\boldsymbol{j} = \boldsymbol{j}_R(r, q, k)$. We can therefore regard $\mathcal{N}^+(k)$ as a Fredholm system of the second kind for the normal components of $(\boldsymbol{\xi}_+, \boldsymbol{\eta}_+)$ in terms of (r, q). In this case, Theorem 6.5 implies that a solution $(\boldsymbol{\xi}_+, \boldsymbol{\eta}_+)$ of THME(k) with vanishing normal components is identically 0 in Ω. Hence $(\boldsymbol{\xi}_-, \boldsymbol{\eta}_-)$ satisfies (7.2), and we can therefore apply the argument leading up to Theorem 7.1 to prove that E_+ is disjoint from the closed upper half-plane. The Fredholm alternative then implies that $\mathcal{N}^+(k)$ is also surjective. The surjectivity of the rows of $\mathcal{T}^+(k)$ is now immediate. $\square$

Remark 7.3. The non-self-adjointness of this BVP places the interior resonances in the lower, nonphysical half-plane. This leads to numerically effective algorithms for solving the THME(k), which do not suffer from the instabilities caused by interior resonances in the physical half-plane.

In the non-simply-connected case we have the following theorem assuring the existence of k-Neumann fields:

THEOREM 7.4 *For $k \in \overline{\mathbb{C}_+}$, the space of k-Neumann fields has dimension exactly $2g$, and the rows of $\mathcal{T}^+(k)$ are surjective.*

PROOF: For $k \in \overline{\mathbb{C}_+}$, the solutions to THME($k$), defined by

$$\mathcal{C}_H = \{(0, 0, \boldsymbol{j}_H, \star_2 \boldsymbol{j}_H) : \boldsymbol{j}_H \in \mathcal{H}^1(\Gamma)\},$$

have a trivial intersection with those defined by data in

$$\mathcal{C}_R = \{(r, q, \boldsymbol{j}_R(r, q, k), \star_2 \boldsymbol{j}_R(r, q, k)) : (r, q) \in \mathcal{M}_{\Gamma,0}\}.$$

The solutions defined by data in $\mathcal{C}_H$ may not themselves be k-Neumann fields. To find solutions in $\mathcal{H}_k(\Omega)$, we first use an element of $\mathcal{C}_H$ to construct a solution $(\boldsymbol{\xi}_{0+}, \boldsymbol{\eta}_{0+})$. If $k \notin E_+$, then we can solve

$$\mathcal{N}^+(k) \begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} i_{\boldsymbol{n}} \boldsymbol{\xi}_{0+} \\ i_{\boldsymbol{n}} \star_3 \boldsymbol{\eta}_{0+} \end{pmatrix} \tag{7.10}$$

and denote the solution of the THME(k) defined by this data in Ω by $(\boldsymbol{\xi}_{1+}, \boldsymbol{\eta}_{1+})$. By Theorem 7.1, the difference

$$(\boldsymbol{\xi}_{N+}, \boldsymbol{\eta}_{N+}) = (\boldsymbol{\xi}_{0+}, \boldsymbol{\eta}_{0+}) - (\boldsymbol{\xi}_{1+}, \boldsymbol{\eta}_{1+}) \tag{7.11}$$

is a nontrivial k-Neumann field. These solutions depend analytically on $k \in \overline{\mathbb{C}_+} \setminus E_+$. As $(\boldsymbol{\xi}_{N+}, \boldsymbol{\eta}_{N+})$ is a nonzero solution to the THME(k) with vanishing normal components, Theorem 6.5 shows that the cohomology class of $\boldsymbol{\xi}_{N+t}$ must be nontrivial. Thus for $k \notin E_+$, $\dim \mathcal{H}_k(\Omega)$ is at least $2g$. On the other hand, the proof of Theorem 6.5 gives the upper bound $\dim \mathcal{H}_k(\Omega) \leq 2g$, thus proving the theorem in this case.

Now suppose that $k_j \in E_+ \cap \overline{\mathbb{C}_+}$. This means that there is a nontrivial, finite-dimensional space of data $V_{k_j} \subset \mathcal{M}_{\Gamma,0}$, which defines $\ker \mathcal{N}^+(k_j)(r, q)$. Let $(\boldsymbol{\xi}_\pm, \boldsymbol{\eta}_\pm)$ denote the solution of the THME(k) defined by a nonzero pair $(r, q) \in V_{k_j}$. The restriction of $\boldsymbol{\xi}_+$ to $T\Gamma$ defines a cohomology class in $H^1_{\mathrm{dR}}(\Gamma)$. If this class is trivial, then Theorem 6.5 implies that the pair $(\boldsymbol{\xi}_+, \boldsymbol{\eta}_+)$ are identically zero. In this case, Theorem 7.1 implies that $\Im k_j < 0$, contradicting the assumption that it lies in $\overline{\mathbb{C}_+}$. This establishes that each nontrivial pair in V_{k_j} defines a nontrivial k_j-Neumann field, thus a subspace of $\mathcal{H}_{k_j}(\Omega)$ of dimension $d = \dim V_{k_j}$.

The Fredholm alternative implies that the equations for the normal components, $\mathcal{N}^+(k_j)(r, q) = (f, g)$, are solvable for pairs $(f, g) \in \mathcal{M}_{\Gamma,0}$ satisfying exactly d linear conditions. This means that within $\mathcal{C}_H$ there is a subspace of dimension at least $2g - d$ for which the normal components can be removed as above. We therefore get another subspace, $U_{k_j} \subset \mathcal{H}_{k_j}(\Omega)$, of dimension at least $2g - d$. As V_{k_j} has a trivial intersection with the data defining U_{k_j}, Theorem 7.1 implies that these two subspaces of $\mathcal{H}_{k_j}(\Omega)$ have a trivial intersection. The lower bound on the dimension of U_{k_j} and the upper bound on $\dim \mathcal{H}_{k_j}(\Omega)$ imply that $\dim U_{k_j} = 2g - d$; this completes the proof that $\dim \mathcal{H}_k(\Omega) = 2g$ for all $k \in \overline{\mathbb{C}_+}$.

For $k \notin E_+$, Theorem 6.5 combined with the fact that $\dim \mathcal{H}_k(\Omega) = 2g$ shows that the rows of $\mathcal{T}^+(k)$ are surjective. If $k \in E_+ \cap \overline{\mathbb{C}_+}$, then the range of $\mathcal{N}^+(k)$

has codimension exactly $d = \dim V_k$. On the other hand, there is a d-dimensional space of data in $\mathcal{H}^1(\Gamma)$ for which the normal components span a complement to that in $\Im\mathcal{N}^+(k)$. Once again we can find an outgoing solution to the THME(k) with any specified normal components. Combined with the fact $\dim \mathcal{H}_k(\Omega) = 2g$, Theorem 6.5 again shows that the rows of $\mathcal{T}^+(k)$ are surjective. $\square$

In the course of this argument we established the following:

COROLLARY 7.5 *For $k \in \overline{\mathbb{C}_+} \setminus \{0\}$, the map from $\mathcal{H}_k(\Omega)$ to $H^1_{\mathrm{dR}}(\Gamma)$ defined by*

$$(\boldsymbol{\xi}_{N+}, \boldsymbol{\eta}_{N+}) \mapsto [\boldsymbol{\xi}_{N+t}]_\Gamma$$

is an isomorphism.

Remark 7.6. It was shown by Picard that for each k with nonnegative real part there are families of interior k-Neumann fields, that is, nontrivial solutions to the THME(k) in D with vanishing normal components; see [25, 26]. For most values of k there is a $2g$-dimensional family. In [17] Kress showed that there is a countable set of positive real numbers $\{k_j\}$, with $k_j \to \infty$, for which there are nontrivial interior k-Neumann fields with vanishing circulations. Does E_+ have a nontrivial intersection with $\Im k \geq 0$? If so, is E_+ related to the set of positive real numbers for which there exist interior k-Neumann fields, with vanishing normal components and circulations?

7.1 Hybrid System Using Forms

The operators defining the tangential component of $\boldsymbol{\xi}_\pm$ are given by

$$\mathcal{T}^\pm_{\boldsymbol{\xi}}(k)\begin{pmatrix} r \\ q \\ \boldsymbol{j} \\ \boldsymbol{m} \end{pmatrix} = \frac{\mp \boldsymbol{m}}{2} + \begin{pmatrix} -K_1 & ikK_{2,t} & -K_4 \end{pmatrix}\begin{pmatrix} r \\ \boldsymbol{j} \\ \boldsymbol{m} \end{pmatrix}. \tag{7.12}$$

If we restrict to $\boldsymbol{j} = \boldsymbol{j}_R(r,q,k)$ and $\boldsymbol{m} = \star_2 \boldsymbol{j}$, then, acting on $(r,q) \in \mathcal{M}_{\Gamma,0}$, the only term of nonnegative order is $-K_1$, which can be expressed as $K_1 r = d_\Gamma G_k r$. Here

$$G_k r(\boldsymbol{x}) = \int_\Gamma g_k(\boldsymbol{x} - \boldsymbol{y}) r(\boldsymbol{y}) dA(\boldsymbol{y}) \tag{7.13}$$

is an operator of order -1. We use $\mathcal{T}^\pm_{\boldsymbol{\xi}}(k)(r,q)$ to denote this operator restricted to this subspace of data.

The hybrid system of integral operators (5.8) is

$$\mathcal{Q}^\pm(k)\begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} -G_0 \star_2 d_\Gamma \mathcal{T}^\pm_{\boldsymbol{\xi}}(k) \\ \mathcal{N}^\pm_{\boldsymbol{\eta}}(k) \end{pmatrix}\begin{pmatrix} r \\ q \end{pmatrix}. \tag{7.14}$$

The range of $\star_2 d_\Gamma \mathcal{T}^\pm_{\boldsymbol{\xi}}(k)$ is contained in the space of functions on Γ with mean zero on every component. Proposition 5.6 holds in the form version as well.

Suppose now that $k \in F_+$ and $\mathcal{Q}^+(k)(r,q) = 0$, with $(r,q) \in \mathcal{M}_{\Gamma,0} \setminus \{0\}$, and let $(\boldsymbol{\xi}_+, \boldsymbol{\eta}_+)$ be the solution to the THME(k) defined by this data. The fact that $\mathcal{Q}^+(k)(r,q) = 0$ implies that

$$d_\Gamma^* \boldsymbol{\xi}_{+t} = 0 \quad \text{and} \quad \boldsymbol{\eta}_+ \restriction_{T\Gamma} = 0; \tag{7.15}$$

the second condition implies that $d_\Gamma \boldsymbol{\xi}_{+t} = 0$ as well. If the cohomology class $[\boldsymbol{\xi}_{+t}]_\Gamma = 0$, then $(\boldsymbol{\xi}_+)_t$ vanishes and Theorem 6.4 implies that $(\boldsymbol{\xi}_+, \boldsymbol{\eta}_+)$ is identically 0. When Γ is simply connected, $H^1_{\mathrm{dR}}(\Gamma) = 0$, and this proves Theorem 5.7, written in terms of forms.

THEOREM 7.7 *If Γ is simply connected, then F_+ is disjoint from the closed upper half-plane. Thus the integral equation*

$$\mathcal{Q}^+(k)\begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} \tag{7.16}$$

provides a unique solution to the scattering problem from a perfect conductor for any k in the closed upper half-plane. Here

$$f = G_0(d_\Gamma^* \boldsymbol{\xi}_t^{\mathrm{in}}), \quad ikh\,dA = d_\Gamma \boldsymbol{\xi}_t^{\mathrm{in}} = ik\boldsymbol{\eta}^{\mathrm{in}} \restriction_{T\Gamma}, \tag{7.17}$$

where $\boldsymbol{\xi}_t^{\mathrm{in}}$ is the tangential component of an incoming electric field, and $\boldsymbol{\eta}^{\mathrm{in}} \restriction_{T\Gamma}$ is the normal component of the incoming magnetic field.

When applying our method in the non-simply-connected case, the following result is useful:

PROPOSITION 7.8 *Suppose that $k \notin E_+ \cup F_+ \cup \{0\}$, and let $\psi \in \mathcal{H}^1(\Gamma)$. The unique outgoing solution to the THME(k) with $\boldsymbol{\xi}_{+t} = \psi$ is defined by data $(r, q, \boldsymbol{j}_H)$ with $\boldsymbol{j}_H \neq 0$.*

PROOF: As $k \notin E_+ \cup \{0\}$, the proof of Theorem 7.4 produces a solution to the THME(k), $(\boldsymbol{\xi}_{N+}, \boldsymbol{\eta}_{N+})$, with vanishing normal components and $[\boldsymbol{\xi}_{N+t}]_\Gamma = [\psi]_\Gamma$. The potentials corresponding to this solution take the form $(r_0, q_0, \boldsymbol{j}_H)$ with $\boldsymbol{j}_H \neq 0$. The condition $[\boldsymbol{\xi}_{N+t}]_\Gamma = [\psi]_\Gamma$ shows that there is a function f, of mean 0 on every component of Γ, satisfying

$$\boldsymbol{\xi}_{N+t} = \psi + d_\Gamma f. \tag{7.18}$$

Since $k \notin F_+$ we can therefore solve the equation

$$\mathcal{Q}^+(k)(r_1, q_1) = (G_0 d_\Gamma^* d_\Gamma f, 0). \tag{7.19}$$

With $(\boldsymbol{\xi}_+, \boldsymbol{\eta}_+)$ the solution to the THME(k) defined in Ω by this data, we see that

$$(\boldsymbol{\xi}_{H+}, \boldsymbol{\eta}_{H+}) = (\boldsymbol{\xi}_{N+}, \boldsymbol{\eta}_{N+}) - (\boldsymbol{\xi}_+, \boldsymbol{\eta}_+) \tag{7.20}$$

satisfies

$$d_\Gamma \boldsymbol{\xi}_{H+t} = d_\Gamma^* \boldsymbol{\xi}_{H+t} = 0 \tag{7.21}$$

and therefore $\boldsymbol{\xi}_{H+t} \in \mathcal{H}^1(\Gamma)$. This solution corresponds to the sources $(r_0 - r_1, q_0 - q_1, \boldsymbol{j}_H)$, with $\boldsymbol{j}_H \neq 0$. Theorem 7.1 then implies that $\boldsymbol{\xi}_{H+} \neq 0$. While it is not clear that $[\boldsymbol{\xi}_{H+t}]_\Gamma = [\psi]_\Gamma$, it follows from (7.21) and Theorem 2.2 that $[\boldsymbol{\xi}_{H+t}]_\Gamma \neq 0$. Thus $\boldsymbol{j}_H \mapsto \boldsymbol{\xi}_{H+t}$ is an injective linear mapping from $\mathcal{H}^1(\Gamma)$ to itself, and therefore an isomorphism. □

Let $\{\psi^1, \dots, \psi^{2g}\}$ be a basis for $\mathcal{H}^1(\Gamma)$. Using the proof of this proposition, along with that of Theorem 7.4, we easily obtain the following corollary:

COROLLARY 7.9 *For $k \notin E_+ \cup F_+ \cup \{0\}$, there are analytic families of solutions*

$$\left\{\left(\boldsymbol{\xi}^1_{+H}(k), \boldsymbol{\eta}^1_{+H}(k)\right), \dots, \left(\boldsymbol{\xi}^{2g}_{+H}(k), \boldsymbol{\eta}^{2g}_{+H}(k)\right)\right\}$$

in Ω to the THME(k), which satisfy

$$\boldsymbol{\xi}^l_{+Ht}(k) = \psi^l \quad \textit{for } l = 1, \dots, 2g. \tag{7.22}$$

One issue left unresolved was the occurrence of interior resonances for $\Im k \geq 0$ and $k \neq 0$. The system of equations (7.14) can be augmented with an additional set of $2g$-conditions to obtain a system that has no null space for $k \in \overline{\mathbb{C}}_+ \setminus \{0\}$. We let $\Psi_H = [\psi^1, \dots, \psi^{2g}]$ denote a basis of harmonic 1-forms on Γ. Using the augmented system, we solve for $(r, q, \boldsymbol{j}_H)$ in one step, and therefore we need to augment the argument list for $\mathcal{Q}^\pm(k)$, replacing the operator in (7.14) with $\mathcal{Q}^\pm(k)(r\; q\; \boldsymbol{j}_H)^\mathsf{T}$. The additional equations are

$$\langle \boldsymbol{\xi}_{\pm t}, \psi^j \rangle = \langle \boldsymbol{\xi}^{\text{in}}_{\pm t}, \psi^j \rangle \quad \text{for } j = 1, \dots, 2g. \tag{7.23}$$

In succinct operator form we denote this by

$$-\left\langle \star_2 \mathcal{T}^\pm_{\boldsymbol{\xi}}(k) \begin{pmatrix} r \\ q \\ \boldsymbol{j}_H \end{pmatrix}, \Psi_H \right\rangle = \langle \boldsymbol{\xi}^{\text{in}}_{\pm t}, \Psi_H \rangle. \tag{7.24}$$

We denote the augmented system by $\mathcal{Q}^\pm_{\text{aug}}(k)$:

$$\mathcal{Q}^\pm_{\text{aug}}(k) \begin{pmatrix} r \\ q \\ \boldsymbol{j}_H \end{pmatrix} = \begin{pmatrix} G_0(d^*_\Gamma \boldsymbol{\xi}^{\text{in}}_t) \\ ik\boldsymbol{\eta}^{\text{in}} \restriction_\Gamma \\ \langle \boldsymbol{\xi}^{\text{in}}_{\pm t}, \Psi_H \rangle \end{pmatrix}. \tag{7.25}$$

THEOREM 7.10 *If $k \in \overline{\mathbb{C}}_+ \setminus \{0\}$ and*

$$\mathcal{Q}^+_{\text{aug}}(k) \begin{pmatrix} r \\ q \\ \boldsymbol{j}_H \end{pmatrix} = 0, \tag{7.26}$$

then $(r, q, \boldsymbol{j}_H) = 0$.

PROOF: The first two equations in $\mathcal{Q}^+_{\text{aug}}(k)$ and the fact that $(\boldsymbol{\xi}_+, \boldsymbol{\eta}_+)$ solves the THME(k) imply that

$$d_\Gamma \boldsymbol{\xi}_{+t} = d^*_\Gamma \boldsymbol{\xi}_{+t} = 0; \tag{7.27}$$

that is, $\boldsymbol{\xi}_{+t}$ is a harmonic 1-form on Γ. The additional conditions in (7.24) imply that the projection of $\boldsymbol{\xi}_{+t}$ onto the harmonic 1-forms is 0 and therefore $\boldsymbol{\xi}_{+t} =$

0. The standard uniqueness theorem for outgoing solutions to the THME(k) then implies that $\boldsymbol{\xi}_+ = 0$ in Ω. Theorem 7.1 then completes the proof. $\square$

This demonstrates that we can use the augmented system to solve the perfect conductor problem for any nonzero frequency with nonnegative imaginary part. This augmented system has conditioning problems as $k \to 0$. Near to $k = 0$, it turns out to be preferable to use the two-step approach, first working with the original unaugmented equations and then correcting the projection of tangential $\boldsymbol{\xi}$-field into the harmonic 1-forms. We consider this briefly in the next section and return to it, in greater detail, in a later publication.

7.2 Low-Frequency Behavior in the Non-Simply-Connected Case

If Γ is not simply connected, then the space of solutions defined by data in $\mathcal{M}_{\Gamma,0}$ converges, as in the simply connected case, to the orthogonal complement of the span of the harmonic Dirichlet and Neumann fields. In this case we also need to consider what happens to the solutions defined by data from $\mathcal{H}^1(\Gamma)$. Using this data we also obtain the harmonic Neumann fields. We consider the two types of data separately, beginning with that from $\mathcal{M}_{\Gamma,0}$.

Theorem 7.1 demonstrates that, for any $k \notin E_+$ with $\Im k \geq 0$, we can solve the boundary value problem

$$
\begin{gathered}
d\boldsymbol{\xi}_+ = ik\boldsymbol{\eta}_+, \quad d^*\boldsymbol{\eta}_+ = -ik\boldsymbol{\xi}_+, \\
i_{\boldsymbol{n}}\boldsymbol{\xi}_+ = f, \quad i_{\boldsymbol{n}}(\star_3\boldsymbol{\eta}_+) = h,
\end{gathered}
\tag{7.28}
$$

for arbitrary (f,h) in $\mathcal{M}_{\Gamma,0}$. Indeed, as $0 \notin E_+$ and the integral equations on Γ are of the second kind and analytic in k, it follows that we can actually solve (7.28) for k in an open neighborhood V of 0. We now discuss what happens to our solutions as k tends to 0 within a relatively compact subset of V. In particular, we would like to characterize exactly which harmonic fields arise as limits of fields of the form given in (6.30), where (r,q) are obtained by solving (4.2), $\boldsymbol{j} = \boldsymbol{j}_R(r,q,k)$ and $\boldsymbol{m} = \star_2\boldsymbol{j}$.

To avoid confusion, we let $(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)$ denote the unique solution to (7.28) for a fixed $(f,h) \in \mathcal{M}_{\Gamma,0}$. As $\boldsymbol{j}_R(r,q,k)$ is $O(k)$, it follows easily that, as k tends to 0, $(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)$ converges to

$$
\boldsymbol{\xi}_0 = d\phi, \quad \boldsymbol{\eta}_0 = d^*\Phi_m. \tag{7.29}
$$

THEOREM 7.11 *The set of limits $(\boldsymbol{\xi}_0, \boldsymbol{\eta}_0)$ for $(f,h) \in \mathcal{M}_{\Gamma,0}$ is the orthogonal complement to the span of both the harmonic Dirichlet and Neumann fields.*

PROOF: As the components decouple at $k = 0$ it suffices to check each separately. Since the Hodge star operator interchanges the solutions, as well as the 1- and 2-form Dirichlet/Neumann fields, we need only check the 1-form case.

First we show that $\boldsymbol{\xi}_0$ is orthogonal to the harmonic Dirichlet fields. These fields are of the form $\boldsymbol{\xi}_d = du$, where u is a harmonic function, constant on each component of Γ. We observe that $u = (|\boldsymbol{x}|^{-1})$ and $\boldsymbol{\xi}_0 = O(|\boldsymbol{x}|^{-2})$, and this justifies the following integration by parts:

$$\begin{aligned} \langle \boldsymbol{\xi}_0, \boldsymbol{\xi}_d \rangle_\Omega 5 &= \langle \boldsymbol{\xi}_0, dr \wedge u \rangle_\Gamma + \langle d^* \boldsymbol{\xi}_0, u \rangle_\Omega \\ &= \langle i_{\boldsymbol{n}} \boldsymbol{\xi}_0, u \rangle_\Gamma = 0. \end{aligned} \tag{7.30}$$

The last equation follows because $i_{\boldsymbol{n}}\boldsymbol{\xi}_0$ has mean zero over every component of Γ, and u is constant on each component.

We now turn to the Neumann fields. A Neumann field $\boldsymbol{\xi}_n$ satisfies $\boldsymbol{\xi}_n = O(|x|^{-2})$. In this instance we use the equation $\boldsymbol{\xi}_0 = d\phi$ to conclude that

$$\begin{aligned} \langle \boldsymbol{\xi}_0, \boldsymbol{\xi}_n \rangle_\Omega &= \langle dr \wedge \phi, \boldsymbol{\xi}_n \rangle_\Gamma + \langle \phi, d^* \boldsymbol{\xi}_n \rangle_\Omega \\ &= \langle \phi, i_{\boldsymbol{n}} \boldsymbol{\xi}_n \rangle_\Gamma = 0. \end{aligned} \tag{7.31}$$

The last equality follows because $i_{\boldsymbol{n}}\boldsymbol{\xi}_n \restriction_\Gamma \equiv 0$ by definition.

An outgoing harmonic 1-form is determined by its normal components along Γ, up to the addition of an arbitrary Neumann field. As the limit $\boldsymbol{\xi}_0$ is required to have mean zero on every component of Γ but is otherwise unrestricted, it follows that every outgoing harmonic 1-form $\boldsymbol{\xi}$ has a unique orthogonal decomposition as

$$\boldsymbol{\xi} = \boldsymbol{\xi}_0 + \boldsymbol{\xi}_d + \boldsymbol{\xi}_n. \tag{7.32}$$

One simply chooses $\boldsymbol{\xi}_d$ so that $i_{\boldsymbol{n}}(\boldsymbol{\xi} - \boldsymbol{\xi}_d)$ has mean zero on every component of Γ. This then uniquely determines $\boldsymbol{\xi}_0$. Recalling that the Dirichlet and Neumann harmonic 1-forms are themselves orthogonal, the Neumann component is then determined by orthogonally projecting $(\boldsymbol{\xi} - \boldsymbol{\xi}_d - \boldsymbol{\xi}_0)$ onto the Neumann fields. This completes the proof of the theorem. □

We now turn to data from $\mathcal{H}^1(\Gamma)$. The first cohomology group of Γ splits into two disjoint subspaces, one is the image of the restriction map $H^1_{\mathrm{dR}}(D) \to \mathcal{H}^1(\Gamma)$, the other the image of $H^1_{\mathrm{dR}}(\Omega) \to \mathcal{H}^1(\Gamma)$. By the Mayer-Vietoris sequence, these restriction maps are injective, and so, by a small abuse of terminology, we may speak of $H^1_{\mathrm{dR}}(\Omega)$ and $H^1_{\mathrm{dR}}(D)$ as subspaces of $H^1_{\mathrm{dR}}(\Gamma)$, and write

$$H^1_{\mathrm{dR}}(\Gamma) = H^1_{\mathrm{dR}}(\Omega) \oplus H^1_{\mathrm{dR}}(D). \tag{7.33}$$

Dually, we have a splitting of the first homology group:

$$H_1(\Gamma) \simeq H_1(\Omega) \oplus H_1(D); \tag{7.34}$$

see [30]. With this notation, $H^1_{\mathrm{dR}}(\Omega)$ is dual to $H_1(\Omega)$, the "A-cycles," shown in Figure 1.1, while $H^1_{\mathrm{dR}}(D)$ is dual to $H_1(D)$, the "B-cycles." We let $\{A_1, \ldots, A_g\}$ denote a basis for the A-cycles and $\{B_1, \ldots, B_g\}$ a the basis of B-cycles. The homology basis can be normalized so that these A- and B-cycles are simple closed curves with the following intersection properties:

$$i(A_j, A_k) = 0, \quad i(A_j, B_k) = \delta_{jk}, \quad i(B_j, B_k) = 0 \quad \text{for } 1 \leq j,k \leq g. \tag{7.35}$$

With these topological preliminaries, we can now turn to the solutions of THME(0).

The solutions to THME(0) are harmonic fields that satisfy the *decoupled* equations

$$d\boldsymbol{\xi}_{\pm} = d^*\boldsymbol{\xi}_{\pm} = 0 \quad \text{and} \quad d\boldsymbol{\eta}_{\pm} = d^*\boldsymbol{\eta}_{\pm} = 0. \tag{7.36}$$

From these equations it is clear that $\boldsymbol{\xi}_{\pm}$ and $\star_3\boldsymbol{\eta}_{\pm}$ are closed 1-forms and therefore define classes in their respective H^1_{dR}-groups. We see that

$$[\boldsymbol{\xi}_{+t}]_\Gamma, [(\star_3\boldsymbol{\eta}_+)_t]_\Gamma \in H^1_{\mathrm{dR}}(\Omega) \quad \text{and} \quad [\boldsymbol{\xi}_{-t}]_\Gamma, [(\star_3\boldsymbol{\eta}_-)_t]_\Gamma \in H^1_{\mathrm{dR}}(D). \tag{7.37}$$

The jump relations show that the solution of the THME(0) defined by the data $(0, 0, \boldsymbol{j}_H)$ satisfies

$$[\boldsymbol{\xi}_{+t} - \boldsymbol{\xi}_{-t}] = -\boldsymbol{j}_H \quad \text{and} \quad [(\star_3\boldsymbol{\eta}_+)_t - (\star_3\boldsymbol{\eta}_-)_t] = -\star_2 \boldsymbol{j}_H. \tag{7.38}$$

According to a theorem of Riemann, we can choose a basis of harmonic 1-forms, $\{\psi^l : l = 1, \dots, 2g\}$ for $H^1_{\mathrm{dR}}(\Gamma)$, so that

$$\psi^{l+g} = \star_2\psi^l \tag{7.39}$$

and

$$\int\limits_{A_k} \psi^l = \delta_{kl}. \tag{7.40}$$

We say that such a basis is adapted to the splitting of $H_1(\Gamma)$ in (7.34). The relations (7.40) show that $\{\psi^{g+1}, \dots \psi^{2g}\}$ lie in the image $H^1_{\mathrm{dR}}(D) \hookrightarrow H^1_{\mathrm{dR}}(\Gamma)$. It is easy to show that they are a basis for this subspace. In general, the elements of $\{\psi^1, \dots \psi^g\}$ have nontrivial projections into both summands in (7.33). Riemann's classic analysis of holomorphic $(1,0)$-forms on algebraic surfaces implies that this is, in general, unavoidable; see [13]. For $l = 1, \dots, g$, define $\psi^l_+ \in H^1_{\mathrm{dR}}(\Omega)$ and $\psi^l_- \in H^1_{\mathrm{dR}}(D)$ so that

$$\psi^l = \psi^l_+ + \psi^l_-. \tag{7.41}$$

For $l = g+1, \dots, 2g$, we have $\psi^l_+ = 0$ and $\psi^l_- = \psi^l$.

We let $\{(\boldsymbol{\xi}^l_{\pm}, \boldsymbol{\eta}^l_{\pm}) : l = 1, \dots, 2g\}$ denote the solutions to the THME(0) defined by the data $(0, 0, \psi^l)$. The relations in (7.37) and the jump relations (7.38) show that

$$[\boldsymbol{\xi}^l_{+t}]_\Gamma + [\psi^l_+]_\Gamma = [\boldsymbol{\xi}^l_{-t}]_\Gamma - [\psi^l_-]_\Gamma = 0, \tag{7.42}$$

because the two sides of these equations belong to disjoint subspaces of $H^1_{\mathrm{dR}}(\Gamma)$. These relations, and analogous ones for the $\boldsymbol{\eta}$-components, easily imply the following result:

PROPOSITION 7.12 *The solutions $\{(\boldsymbol{\xi}^l_{\pm}, \boldsymbol{\eta}^l_{\pm})\}$ defined by the adapted basis of harmonic forms $\{\psi^l\}$ satisfy the following*:

(i) *For $l = 1, \dots, g$, the restrictions $[\boldsymbol{\xi}^l_{+t}]_\Gamma = -[\psi^l_+]_\Gamma$ span $H^1_{\mathrm{dR}}(\Omega) \subset H^1_{\mathrm{dR}}(\Gamma)$, while the restrictions $[(\star_3\boldsymbol{\eta}^l_+)_t]_\Gamma = 0$.*

(ii) *For* $l = 1, \dots, g$, *the restrictions* $[(\star_3 \boldsymbol{\eta}_+^{g+l})_t]_\Gamma = [\psi_+^l]_\Gamma$ *span* $H^1_{\mathrm{dR}}(\Omega) \subset H^1_{\mathrm{dR}}(\Gamma)$, *while the restrictions* $[\boldsymbol{\xi}_{+t}^{g+l}]_\Gamma = 0$.

We now recall the basis of k-Neumann fields, $N_k = \{(\boldsymbol{\xi}^l_{N+}(k), \boldsymbol{\eta}^l_{N+}(k)) : l = 1, \dots, 2g\}$, constructed in the proof of Theorem 7.4. This is an analytic family in a neighborhood of 0, as it only requires the solvability of the normal equations. We now assume that we define these fields, using a basis $\{\psi^l : l = 1, \dots, 2g\}$ of $\mathcal{H}^1(\Gamma)$, which is adapted to the splitting in (7.34). Proposition 7.12 shows that at $k = 0$ the fields in N_k continue to span a $2g$-dimensional vector space of solutions to the THME(0), and that $\{\boldsymbol{\xi}^l_{N+}(0) : l = 1, \dots, g\}$ are a basis for the space of outgoing, harmonic 1-forms, with vanishing normal component along $b\Omega$. As $\boldsymbol{j}_R(r, q, 0) = 0$, we see that these fields are defined by data of the form (r^l, q^l, ψ^l) and therefore continue to satisfy

$$[\boldsymbol{\xi}^l_{N+t}(0)]_\Gamma = -[\psi^l_+]_\Gamma, \quad [(\star_3 \boldsymbol{\eta}^l_{N+}(0))_t]_\Gamma = 0, \tag{7.43}$$

and

$$[\boldsymbol{\xi}^{g+l}_{N+t}(0)]_\Gamma = 0, \quad [(\star_3 \boldsymbol{\eta}^{g+l}_{N+}(0))_t]_\Gamma = [\psi^l_+]_\Gamma. \tag{7.44}$$

Here $l = 1, \dots, g$.

These fields are outgoing, harmonic 1-forms, with vanishing normal components along $b\Omega$; hence there must be constants $\{a_1, \dots, a_g\}$ so that

$$\boldsymbol{\xi}^{l+g}_{N+}(0) = \sum_{m=1}^{g} a_m \boldsymbol{\xi}^m_{N+}(0). \tag{7.45}$$

The first equation in (7.44) implies that all the coefficients are 0. This and an identical argument for $\{\star_3 \boldsymbol{\eta}^l_{N+}(0)\}$ prove that the basis N_k reduces at $k = 0$ to a basis of the form

$$\left\{\left(\boldsymbol{\xi}^l_{N+}(0), 0\right), \left(0, \boldsymbol{\eta}^{l+g}_{N+}(0)\right) : l = 1, \dots, g\right\}. \tag{7.46}$$

We summarize these results in a theorem.

THEOREM 7.13 *There is an open neighborhood U of* $0 \in \mathbb{C}$, *and* $2g$ *analytic families of outgoing solutions* $\{(\boldsymbol{\xi}^l_{N+}(k), \boldsymbol{\eta}^l_{N+}(k)) : l = 1, \dots, 2g\}$ *to the THME*(k), *which, for each* $k \in U$, *are a basis for the* k*-Neumann fields* $\mathcal{H}_k(\Omega)$. *At* $k = 0$ *the* $\boldsymbol{\xi}$*- and* $\boldsymbol{\eta}$*-components decouple and satisfy* (7.46).

Theorems 7.11 and 7.13 give a clear picture of the behavior of the space of outgoing solutions to the THME(k) in a neighborhood of 0, defined by the representation (6.30). They show that, in a reasonable sense, this representation does not suffer from low-frequency breakdown. Let $U \subset \mathbb{C}$ be a neighborhood of 0, and $\{\boldsymbol{\alpha}(k) : k \in U\}$ a continuous family of 1-forms defined on Γ. If $\boldsymbol{\alpha}$ is orthogonal to $\mathcal{H}^1(\Gamma)$ and $d_\Gamma \boldsymbol{\alpha}(k)/k$ has a limit as k tends to 0, then it is clear that the hybrid system provides a continuous family of solutions, in a neighborhood of 0, to the THME(k) with $\boldsymbol{\xi}_{+t}(k) = \boldsymbol{\alpha}(k)$. In a subsequent publication we will consider conditions on the projection of $\boldsymbol{\alpha}(k)$ into $\mathcal{H}^1(\Gamma)$ that are needed to conclude the existence of such a continuous family of solutions.

8 Normal Component Equations on the Unit Sphere

In this and the following section we determine the exact form of the systems of Fredholm equations derived above for the special case of the unit sphere in $\mathbb{R}^3$. We make extensive usage of spherical harmonics and "vector" spherical harmonics in the exterior form representation. As this is not standard, these formulæ are derived in the Appendix. The equations decouple and very nicely illustrate the general properties described above. As the equations for the normal components are a bit simpler, we begin with them.

The integral equations for the normal components of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ can be solved simply and explicitly when Γ is the unit sphere centered at 0. This reveals the close connection between our equations and the Mie-Debye solution. We are representing $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in terms of the potentials $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}_m$, ϕ, and Φ_m, with $\boldsymbol{j}$ a 1-form on S_1^2 and $\boldsymbol{m} = \star_2 \boldsymbol{j}$. The Debye sources r and q satisfy

$$ikr\,dA = d_{S_1^2} \star_2 \boldsymbol{j} \quad \text{and} \quad ikq\,dA = d_{S_1^2} \boldsymbol{j}\,. \tag{8.1}$$

If we assume that

$$\begin{aligned} r &= \sum_{lm} a_{lm} Y_l^m, \quad q = \sum_{lm} b_{lm} Y_l^m, \\ \boldsymbol{j} &= \sum_{lm} \left[\alpha_{lm} d_{S_1^2} Y_l^m + \beta_{lm} \star_2 d_{S_1^2} Y_l^m\right], \end{aligned} \tag{8.2}$$

then (8.1) implies that

$$-l(l+1)\alpha_{lm} = ika_{lm} \quad \text{and} \quad -l(l+1)\beta_{lm} = ikb_{lm}. \tag{8.3}$$

Suppose that the normal components of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are represented in terms of spherical harmonics by

$$i_{\boldsymbol{n}}\boldsymbol{\xi} = \sum_{lm} c_{lm} Y_l^m \quad \text{and} \quad i_{\boldsymbol{n}} \star_3 \boldsymbol{\eta} = \sum_{lm} d_{lm} Y_l^m. \tag{8.4}$$

Using the results of Propositions A.2 and A.3 in the Appendix, we see that the integral equations in (4.2) for the different spherical harmonic components decouple. The equation for the coefficient of the lm-component of $i_{\boldsymbol{n}}\boldsymbol{\xi}$ becomes

$$\begin{aligned} &c_{lm} Y_l^m = \\ &\quad i_{\partial_r}\big[ik(\alpha_{lm} G_k[(d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}] + \beta_{lm} G_k[(\star_2 d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}]) - dG_k(a_{lm} Y_l^m) \\ &\qquad - \star_3 d(\alpha_{lm} G_k[(\star_2 d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}] - \beta_{lm} G_k[(d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}])\big] \end{aligned} \tag{8.5}$$

From these results we also see that this equation reduces to

$$\begin{aligned} c_{lm} = ik\alpha_{lm}\left(\frac{ikl(l+1)}{2l+1}\right)\left[j_{l-1}(k)h^{(1)}_{l-1}(k) - j_{l+1}(k)h^{(1)}_{l+1}(k)\right] \\ - ik^2 a_{lm} j_l(k)\partial_k h^{(1)}_l(k) + ikl(l+1)\alpha_{lm} j_l(k)h^{(1)}_l(k). \end{aligned} \tag{8.6}$$

Standard recurrence relations for the spherical Bessel functions imply that

$$\begin{aligned} \frac{k\left(j_{l-1}(k)h^{(1)}_{l-1}(k) - j_{l+1}(k)h^{(1)}_{l+1}(k)\right)}{2l+1} = \\ \frac{j_l(k)h^{(1)}_l(k)}{k} + j_l(k)\partial_k h^{(1)}_l(k) + \partial_k j_l(k)h^{(1)}_l(k); \end{aligned} \tag{8.7}$$

see [15]. Using this identity and the relations in (8.3), we obtain

$$c_{lm} = a_{lm}kh^{(1)}_l(k)(ij_l(k) + ikj'_l(k) + kj_l(k)). \tag{8.8}$$

We define the function

$$m_n(k,l) = kh^{(1)}_l(k)((i+k)j_l(k) + ikj'_l(k)). \tag{8.9}$$

An essentially identical sequence of steps leads to the relations

$$d_{lm} = -m_n(k,l)b_{lm}. \tag{8.10}$$

The diagonal entries of the block diagonal matrix we need to invert to solve the normal component problem for the unit sphere at frequency k are simply $\{m_n(k,l) : l = 1, 2, \dots\}$.

Remark 8.1. The connection to classical Debye theory is now easy to establish. If we expand the Debye potentials u and v in (0.7) as

$$v(r,\theta,\phi) = \sum_{l,m} a_{lm}h^{(1)}_l(kr)Y^m_l, \quad u(r,\theta,\phi) = \sum_{l,m} b_{lm}h^{(1)}_l(kr)Y^m_l,$$

then a straightforward calculation [24] shows that (in terms of the normal components)

$$a_{lm} = \frac{1}{l(l+1)}c_{lm}, \quad b_{lm} = -\frac{1}{l(l+1)}d_{lm}.$$

Thus, our generalized Debye sources, defined only on the surface, are analogous (but not equivalent) to the restrictions of the Debye potentials to the sphere. The classical Debye approach requires that the potentials themselves (defined in $\mathbb{R}^3 \setminus \Gamma$) be expanded in surface harmonics, preventing the approach from being extensible to arbitrary geometry.

We apply the Wronskian identity

$$j_l(k)\partial h^{(1)}_l(k) - \partial_k j_l(k)h^{(1)}_l(k) = \frac{i}{k^2} \tag{8.11}$$

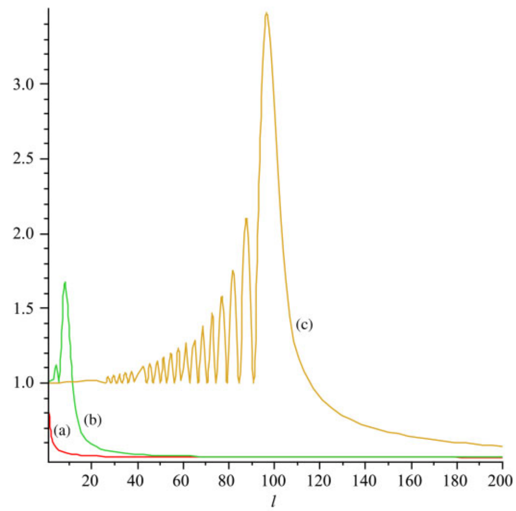


FIGURE 8.1. Plots of (a) $|m_n(1,l)|$, (b) $|m_n(10,l)|$, and (c) $|m_n(100,l)|$.

to find that

$$m_n(k,l) = \big(1 + ikj_l(k)h_l^{(1)}(k) + ik^2 j_l(k)\partial_k h_l^{(1)}(k) + k^2 j_l(k)h_l^{(1)}(k)\big). \tag{8.12}$$

Applying standard asymptotic formulæ for j_l and $h_l^{(1)}$ to this representation shows that, for a fixed k with nonnegative real part, we have

$$m_n(k,l) \sim \frac{1}{2} - \frac{i}{2l+1} + O(l^{-2}). \tag{8.13}$$

This agrees with the fact that the integral equation for $\boldsymbol{i_n \xi}$ is of the form $\frac{1}{2} + K(k)$, where K is compact. It is also the case that

$$m_n(0,l) = \frac{1}{2} - \frac{i}{2l+1}, \tag{8.14}$$

which shows that these equations do not exhibit low-frequency breakdown. For integral l and k along the real axis we have

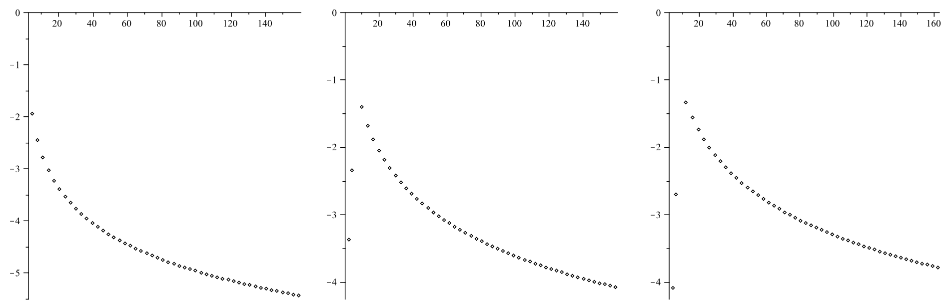
$$m_n(k,l) \sim 1 + O(k^{-1}); \tag{8.15}$$

as $\Im k$ tends to infinity, we have

$$m_n(k,l) \sim 1 + O(k^{-1}). \tag{8.16}$$

Figure 8.1 shows plots of $\{|m_n(k,l)| : k = 1, 10, 100\}$. The condition number increases with the frequency k.

For fixed l the solutions of $m_n(k,l) = 0$ have $\Im k < 0$. If l is fixed, then the imaginary parts of the roots of $m_n(k,l) = 0$ decrease in proportion to minus the



(a) Zeros of $m_n(k,1)$. (b) Zeros of $m_n(k,5)$. (c) Zeros of $m_n(k,7)$.

FIGURE 8.2. Graphs of the first 50 zeros of the multipliers $m_n(k,l)$ for $l = 1, 5, 7$.

log of the real part,

$$\Im k \propto -\frac{1}{2} \log \Re k. \tag{8.17}$$

The first 50 zeros of $m_n(k,1)$, $m_n(k,5)$, and $m_n(k,7)$ are shown in Figure 8.2.

Using the three-term recurrence relations for spherical Bessel functions $\{z_l\}$, we easily obtain the functional equation

$$z_l(-\bar{k}) = (-1)^l \overline{z_l(k)}; \tag{8.18}$$

using this identity and (8.9) it is not difficult to show that

$$m_n(-\bar{k}, l) = \overline{m_n(k,l)}. \tag{8.19}$$

The function $h_l^{(1)}$ can be factored as

$$h_l^{(1)}(k) = p_l(k) \frac{e^{ik}}{k^{l+1}}, \tag{8.20}$$

where p_l is a polynomial of degree l. Thus the multiplier takes the form

$$\begin{aligned} m_n(k,l) &= p_l(k) e^{ik} \left(\frac{(i+k) j_l(k) + ik j_l'(k)}{k^l} \right) \\ &= p_l(k) e^{ik} \left(\frac{(k - il) j_l(k) + ik j_{l-1}(k)}{k^l} \right). \end{aligned} \tag{8.21}$$

It is easy to see that the numerator has a zero of order l at $k = 0$, and therefore the quotient is regular and nonvanishing there. The polynomial contributes l roots; the symmetry (8.19) shows that, when l is odd, one root lies on the negative imaginary axis.

Remark 8.2. The proof of Theorem 7.1 shows that the frequencies k, with $\Im k < 0$, for which equation (4.2) has a nontrivial null space, coincides with the eigenvalues of the *interior* boundary value problem for Maxwell's equations defined by

$$\boldsymbol{\xi}_- \restriction_\Gamma = i_{\boldsymbol{n}} \boldsymbol{\eta}_- \restriction_\Gamma . \tag{8.22}$$

Of course, there are no eigenvalues or resonances with $\Im k \geq 0$. Calculations like those above, though simpler, show that this boundary condition holds for the vector spherical harmonics of order l provided

$$(i+k) j_l(k) + ikj_l'(k) = 0 \quad \text{with } k \neq 0. \tag{8.23}$$

From (8.21), we see that the left-hand side of (8.23) is a factor of $m_n(k,l)$. Thus the eigenvalues of the interior problem are a subset of the resonances of the exterior representation. These eigenvalues are the "nonphysical" interior resonances connected with our representation (6.30) of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in terms of potentials. They are familiar from the EFIE and MFIE representation, but shifted to the lower half-plane, where they do no serious harm. The roots of $h_l^{(1)}(k)$ are known to be related to scattering resonances for scattering off of a conducting sphere.

9 Hybrid Equations on the Unit Sphere

To find the precise form of the hybrid operator $\mathcal{Q}^+(k)$ on the unit sphere, we only need to work out

$$-G_0 d^*_{S_1^2} \star_2 \mathcal{T}^+_{\boldsymbol{\xi}}(k).$$

The normal equation is given by (8.10). As in the previous section, we represent $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in terms of the potentials $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}_m$, ϕ, and Φ_m, with $\boldsymbol{j} = \boldsymbol{j}_R(r,q,k)$ a 1-form on S_1^2 and $\boldsymbol{m} = \star_2 \boldsymbol{j}$. The generalized Debye sources r and q satisfy (8.2) and (8.3). The $\boldsymbol{\xi}$-field is given in terms of the potentials by

$$\boldsymbol{\xi} = [ikG_k \boldsymbol{j} \cdot d\boldsymbol{x} - dG_k r - \star_3 dG_k \star_2 \boldsymbol{j} \cdot d\boldsymbol{x}]. \tag{9.1}$$

Using the expressions for r and $\boldsymbol{j}$ in terms of spherical and vector spherical harmonics, respectively, we see that the tangential components of $\boldsymbol{\xi}_+$ modulo $\ker d^*_{S_1^2}$ are given by

$$\begin{aligned} &\boldsymbol{\xi}_{+t} \operatorname{mod} \ker d^*_{S_1^2} = \\ &\quad \sum_{l,m} d_{S_1^2} Y_l^m \bigg[\left(\frac{-k^2 \alpha_{lm}}{2l+1} \right) \big[(l+1) j_{l-1}(k) h_{l-1}^{(1)}(k) + l j_{l+1} h_{l+1}^{(1)}(k) \big] \\ &\qquad - ika_{lm} j_l(k) h_l^{(1)}(k) + ik\alpha_{lm} j_l(k) \big[h_l^{(1)}(k) + k\partial_k h_l^{(1)}(k) \big] \bigg]. \end{aligned} \tag{9.2}$$

Using (8.3) and the identity

$$G_0 d^*_{S_1^2} [d_{S_1^2} Y_l^m] = \frac{l(l+1)}{2l+1} Y_m^l , \tag{9.3}$$

we easily obtain that

$$G_0 d^*_{S^2_1} \mathcal{T}^+_{\boldsymbol{\xi}}(k) \begin{pmatrix} r \\ q \end{pmatrix} = \sum_{l,m} \left(\frac{a_{lm} Y^m_l}{2l+1} \right) \Big[-ikl(l+1) j_l(k) h^{(1)}_l(k) + k^2 j_l(k) \big[h^{(1)}_l(k) + k \partial_k h^{(1)}_l(k) \big] + \frac{ik^3 [(l+1) j_{l-1}(k) h^{(1)}_{l-1}(k) + l j_{l+1} h^{(1)}_{l+1}(k)]}{2l+1} \Big]. \tag{9.4}$$

As with the system of normal equations, the hybrid equations are decoupled, providing one equation for the coefficients of r and one for the coefficients of q. The only term on the right-hand side of (9.4) that is not $O(l^{-1})$ is

$$\frac{-ikl(l+1) j_l(k) h^{(1)}_l(k)}{2l+1} = \frac{-1}{4} + O(l^{-1}), \tag{9.5}$$

in agreement with (5.10). We use the identity

$$\partial_k \big(k h^{(1)}_l(k) \big) = k h^{(1)}_{l-1}(k) - l h^{(1)}_l(k) \tag{9.6}$$

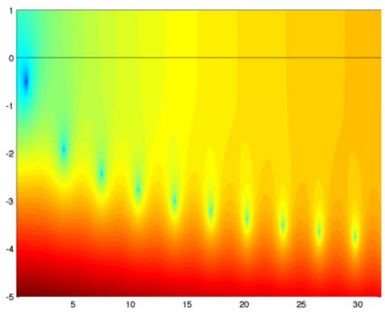
to remove the derivative from (9.4) and define the multiplier for the tangential equation

$$m_t(k, l) = \left(\frac{-k}{2l+1} \right) \Big[il(l+1) j_l(k) h^{(1)}_l(k) - k j_l(k) [k h^{(1)}_{l-1}(k) - l h^{(1)}_l(k)] - \frac{ik^2 [(l+1) j_{l-1}(k) h^{(1)}_{l-1}(k) + l j_{l+1}(k) h^{(1)}_{l+1}(k)]}{2l+1} \Big]. \tag{9.7}$$

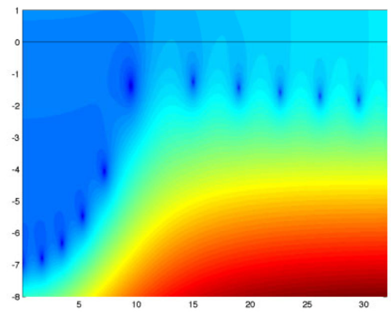
This multiplier behaves much like the multiplier $m_n(k, l)$ found for the normal equations. For fixed l its roots, as a function of k, lie in the lower half-plane. The plots in Figure 9.1 show contours of $\log |m_t(k, l)|$ for $l = 1, 10, 20$. The x-axis is shown as a black horizontal line. They clearly show that the zeros lie in the lower half-plane and indicate the moderate behavior of the multiplier in the upper half-plane.

If the incoming tangential data are given by

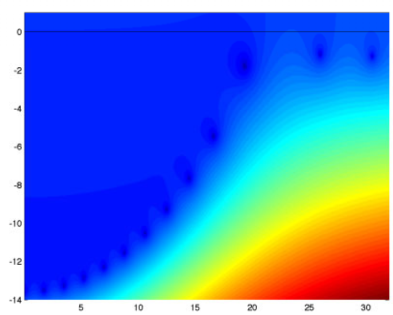
$$\boldsymbol{\xi}^{\text{in}}_t = \sum_{l,m} \left[p_{lm} \frac{d_{S^2_1} Y^m_l}{\sqrt{l(l+1)}} + q_{lm} \frac{\star_2 d_{S^2_1} Y^m_l}{\sqrt{l(l+1)}} \right] \tag{9.8}$$



(a) Contour plot of of $\log |m_t(k, 1)|$.



(b) Contour plot of $\log |m_t(k, 10)|$.



(c) Contour plot of of $\log |m_t(k, 25)|$.

FIGURE 9.1. Plots of $\log |m_t(k, 1)|$, $\log |m_t(k, 10)|$, $\log |m_t(k, 25)|$. The black horizontal line indicates the x-axis. The zeros are located near the deep blue dots.

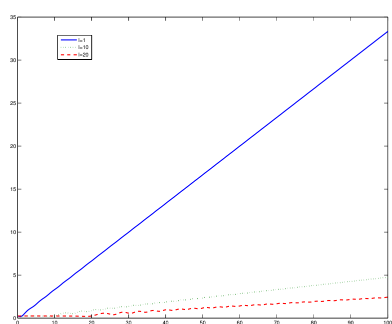


FIGURE 9.2. Plots of $|m_t(k, 1)|$, $|m_t(k, 10)|$, and $|m_t(k, 20)|$ for $k \in [0, 100]$.

(here we used the normalized basis elements), then

$$
\begin{aligned}
i_{\boldsymbol{n}} \star_3 \boldsymbol{\eta}^{\text{in}} = \frac{\star_2 d_{S_1^2} \boldsymbol{\xi}_t^{\text{in}}}{ik} &= \sum_{l,m} q_{lm} \frac{\star_2 d_{S_1^2} \star_2 d_{S_1^2} Y_l^m}{ik\sqrt{l(l+1)}} \\
&= -\frac{1}{ik} \sum_{l,m} \sqrt{l(l+1)} q_{lm} Y_l^m .
\end{aligned}
\tag{9.9}
$$

To find the tangential data for the hybrid system, we apply $G_0 d^*_{S_1^2}$ to $\boldsymbol{\xi}_t^{\text{in}}$ obtaining:

$$
G_0 d^*_{S_1^2} \boldsymbol{\xi}_t^{\text{in}} = \sum_{l,m} \frac{\sqrt{l(l+1)}}{2l+1} p_{lm} Y_l^m . \tag{9.10}
$$

For the unit sphere, the hybrid equations are therefore

$$
m_n(k,l) b_{lm} = \frac{\sqrt{l(l+1)} q_{lm}}{ik}, \quad m_t(k,l) a_{lm} = \frac{\sqrt{l(l+1)}}{2l+1} p_{lm} . \tag{9.11}
$$

These equations display a mild sort of low-frequency breakdown in that the coefficients of the normal data, $\{\sqrt{l(l+1)} q_{lm}\}$, must be uniformly $O(\omega)$ in order for this system of equations to be stable. Of course, the incoming data $(\boldsymbol{\xi}_+^{\text{in}}, \boldsymbol{\eta}_+^{\text{in}})$ is assumed to be a solution of the THME(k), so these estimates should automatically hold. Indeed if $\boldsymbol{\eta}_+^{\text{in}}$ is given, then there is no need to differentiate $\boldsymbol{\xi}_+^{\text{in}}$ and divide by k to find the data for the normal equation.

The use of G_0 in the preconditioner also leads to growth in the multiplier $m_t(k,l)$ as k increases for fixed l. Figure 9.2 shows, for real $k \in [0,100]$, $|m_t(k,1)|$, $|m_t(k,10)|$, and $|m_t(k,20)|$. In the interval $0 < k < l$ these functions oscillate around a small nonzero value. When k exceeds l, these functions show linear growth. Replacing G_0 with something like $G_{i|k|}$ should fix this problem.

10 Conclusions

In this paper, we have developed a new representation for solutions of the time-harmonic Maxwell equations, exterior to closed surfaces, based on two scalar densities. In the zero frequency limit, these densities are uncoupled and correspond to electric and magnetic charge. At nonzero frequency, however, they do not correspond directly to physical variables. They are simply used to construct electric and magnetic currents, after which the classical scalar and vector potentials and antipotentials are employed (in the usual Lorenz gauge). Because of the close connection to the Lorenz-Debye-Mie formalism when the analysis is restricted to the unit sphere, we refer to our unknowns as *generalized Debye sources*. The natural boundary data for our unknowns are the normal components of the electric and magnetic field, and we have provided a detailed uniqueness theory for this boundary value problem for boundary surfaces of arbitrary genus (Theorems 4.1, 4.3, and 6.5). In the course of this analysis, we have given a new proof of the existence (in the non-simply-connected case) of families of nontrivial solutions with zero boundary data, which we refer to as k-Neumann fields. They generalize, to nonzero wave numbers, the classical harmonic Neumann fields (Theorem 7.4). It should be noted that, unlike the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$, the normal components of solutions to the THME(k) do not have to satisfy additional compatability conditions at $k = 0$, and this explains, in part, the good behavior of the representation in terms of Debye sources as $k \to 0$.

We have also introduced new Fredholm integral equations of the second kind for scattering from a perfect electrical conductor and have shown that it is invertible for all nonzero wave numbers k in the closed upper half-plane. There is a natural extension of the approach to the case of a dielectric interface, which will be reported at a later date.

The work begun here gives rise to a new set of analytic and computational issues. In order to use the Debye sources as unknowns, one needs an efficient and accurate method for inverting the surface Laplacian. For surfaces Γ of genus $g > 0$, we also need to be able to efficiently construct a basis for the harmonic forms $\mathcal{H}^1(\Gamma)$. Finally, additional work is required to extend our approach to open surfaces (see, for example, [16]). These arise as common and important idealizations in the analysis of thin plates, cylindrical conductors, and metallized surface patches in radar, medical imaging, chip design, and remote sensing applications.

Appendix: Exterior Forms and Vector Spherical Harmonics

The domains defined in $\mathbb{R}^3$ as complements of a round sphere are very important in applications. They also provide a context where the integral equations defined in the earlier sections can be diagonalized and solved explicitly in terms of classical special functions. In this appendix we give a brief treatment of the theory of vector spherical harmonics in the exterior form representation. A classical treatment, in terms of vectors fields, is given in sections 9.6 and 9.7 of [15]. But for the classical theory of (scalar) spherical harmonics (which can also be found in Jackson), our discussion is essentially self-contained.

We begin with the relationship between the (negative) Laplace-Beltrami operators in $\mathbb{R}^3$ and on the unit sphere $S^2_1 \subset \mathbb{R}^3$. Recall that on any Riemannian manifold (X, g), the Laplace operator on k-forms is given by $\Delta^X_k = -(d^*d + dd^*)$, where $*$ is defined by g. The following is simply the usual change of variables formula for spherical polar coordinates.

PROPOSITION A.1 *Let $r^2 = x_1^2 + x_2^2 + x_3^2$. The scalar Laplace operator on $\mathbb{R}^3$ can be expressed as*

$$\Delta_0^{\mathbb{R}^3} = \frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}\Delta_0^{S_1^2} \quad \textit{where } \partial_r = \frac{x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}}{r}. \tag{A.1}$$

Let $\boldsymbol{\alpha}$ *be a* 1*-form on* $\mathbb{R}^3$ *such that* $i_{\partial_r}\boldsymbol{\alpha} = 0$; *then*

$$\Delta_1^{\mathbb{R}^3}\boldsymbol{\alpha} = \frac{1}{r^2}\Delta_1^{S_1^2}\boldsymbol{\alpha} + L_r\boldsymbol{\alpha} + \frac{1}{r^2}\big(d^*_{S_1^2}\boldsymbol{\alpha}\big)dr, \tag{A.2}$$

where

$$L_r\boldsymbol{\alpha} = i_{r^2\partial_r}d(i_{r^{-2}\partial_r}d\boldsymbol{\alpha}) + \frac{2}{r^2}\boldsymbol{\alpha}. \tag{A.3}$$

This formula follows by a calculation using a local coframe field. If we express $\boldsymbol{\alpha} = \sum \alpha_j\, dx_j$, then

$$L_r\boldsymbol{\alpha} = \sum_{j=1}^{3}(\partial_r^2\alpha_j)dx_j. \tag{A.4}$$

While (A.2) is a good deal more complicated than (A.1), it allows for a careful analysis of the eigenforms of $\Delta_1^{S_1^2}$.

Let $\mathcal{E}_l^0$ denote the linear space of scalar eigenfunctions on S_1^2 satisfying

$$\Delta_0^{S_1^2} f = -l(l+1)f. \tag{A.5}$$

These spaces are represented in terms of classical spherical harmonics by

$$\mathcal{E}_l^0 = \operatorname{span}\{Y_l^m : m = -l, \dots, l\}. \tag{A.6}$$

A basis of eigenforms of $\Delta_2^{S_1^2}$, with eigenvalue $-l(l+1)$, is given by

$$\{\star_2 Y_l^m = Y_l^m\, dA : m = -l \dots l\}.$$

The fact that $H^1(S^2;\mathbb{R}) = 0$ and the Hodge theorem imply that the 1-forms on S_1^2 are the L^2-orthogonal direct sum

$$\mathcal{C}^\infty(S_1^2;\Lambda^1) = d_{S_1^2}\mathcal{C}^\infty(S_1^2;\Lambda^0) \oplus d^*_{S_1^2}\mathcal{C}^\infty(S_1^2;\Lambda^2). \tag{A.7}$$

This observation coupled with the fact that d and d^* commute with Δ imply that the eigenspace of $\Delta_1^{S_1^2}$, with eigenvalue $-l(l+1)$, is given by

$$\mathcal{E}_l^1 = \operatorname{span}\big[\{d_{S_1^2}Y_l^m : m = -l, \dots, l\} \oplus \{\star_2 d_{S_1^2}Y_l^m : m = -l, \dots, l\}\big]. \tag{A.8}$$

From this representation, the classical orthogonality relations are quite easy:

$$\begin{aligned}
\langle d_{S_1^2}Y_l^m, d_{S_1^2}Y_{l'}^{m'}\rangle &= \langle d^*_{S_1^2}d_{S_1^2}Y_l^m, Y_{l'}^{m'}\rangle = \delta_{ll'}\delta^{mm'}l(l+1),\\
\langle \star_2 d_{S_1^2}Y_l^m, \star_2 d_{S_1^2}Y_{l'}^{m'}\rangle &= \langle d_{S_1^2}Y_l^m, d_{S_1^2}Y_{l'}^{m'}\rangle = \delta_{ll'}\delta^{mm'}l(l+1),\\
\langle d_{S_1^2}Y_l^m, \star_2 d_{S_1^2}Y_{l'}^{m'}\rangle &= \langle Y_l^m, d^2_{S_1^2}Y_{l'}^{m'}\rangle = 0.
\end{aligned} \tag{A.9}$$

The second line derives from the fact that $\star_2$ is an orthogonal transformation, and the last relation follows from Stokes theorem.

The eigenforms $\star_2 d_{S_1^2}Y_l^m$ are divergence free. We extend them to $\mathbb{R}^3$ so they annihilate ∂_r, and express them in the form $\star_2 dY_l^m = a_1dx_1 + a_2dx_2 + a_3dx_3$.

Here the coefficients are extended to be homogeneous of degree 0. It follows from (A.2) and the equation

$$\Delta_1^{\mathbb{R}^3}(a_1 dx_1 + a_2 dx_2 + a_3 dx_3) = \big(\Delta_0^{\mathbb{R}^3} a_1\big)dx_1 + \big(\Delta_0^{\mathbb{R}^3} a_2\big)dx_2 + \big(\Delta_0^{\mathbb{R}^3} a_3\big)dx_3 \tag{A.10}$$

that, where $r = 1$, we have

$$\Delta_0^{S_1^2} a_j = -l(l+1)a_j. \tag{A.11}$$

In other words, the coefficients of $\star_2 d_{S_1^2} Y_l^m$ lie in $\mathcal{E}_l^0$. These eigenforms correspond to the classical eigenfields of the form $\{\boldsymbol{r} \times \nabla Y_l^m\}$.

The members of the other family, $\{d_{S_1^2} Y_l^m\}$, which corresponds to $\{\boldsymbol{r} \times (\boldsymbol{r} \times \nabla Y_l^m)\}$, are not divergence free, and their coefficients with respect to dx_j lie in $\mathcal{E}_{l-1}^0 \oplus \mathcal{E}_{l+1}^0$. These coefficients are easily found; if we think of Y_l^m as a homogeneous function of degree 0 on $\mathbb{R}^3$, then $i_{\partial_r} d_{\mathbb{R}^3} Y_l^m = 0$,

$$d_{S_1^2} Y_l^m = \sum_{j=1}^{3} \frac{\partial Y_l^m}{\partial x_j} dx_j \restriction_{S_1^2}. \tag{A.12}$$

In order to determine the action of the Green's function on the coefficients of these forms, we need to represent them in terms of spherical harmonics. Let $U_l^m = r^l Y_l^m$. This is a homogeneous harmonic polynomial of degree l. We see that

$$\frac{\partial U_l^m}{\partial x_j} = r^l \frac{\partial Y_l^m}{\partial x_j} + l \frac{x_j}{r^2} U_l^m. \tag{A.13}$$

We apply the Laplace operator to $x_j U_l^m$ to obtain

$$\Delta_0^{\mathbb{R}^3}(x_j U_l^m) = 2\frac{\partial U_l^m}{\partial x_j} \tag{A.14}$$

and therefore

$$x_j U_l^m = u_{l+}^{mj} + r^2 u_{l-}^{mj}. \tag{A.15}$$

Here u_{l+}^{mj}, u_{l-}^{mj} are homogeneous harmonic polynomials of degrees $l+1$ and $l-1$, respectively. Once again applying the Laplace operator to this relation, we see that

$$u_{l-}^{mj} = \frac{1}{2l+1} \frac{\partial U_l^m}{\partial x_j} \tag{A.16}$$

and therefore

$$\sum_{j=1}^{3} u_{l+}^{mj}\, dx_j = U_l^m r\, dr - \frac{r^2}{2l+1}\, dU_l^m. \tag{A.17}$$

Using the homogeneity we also see that

$$\sum_{j=1}^{3} x_j u_{l+}^{mj} = \frac{l+1}{2l+1} r^2 U_l^m. \tag{A.18}$$

Restricting to $r = 1$ gives

$$\frac{\partial Y_l^m}{\partial x_j} \restriction_{r=1} = \frac{l+1}{2l+1} \frac{\partial U_l^m}{\partial x_j} \restriction_{r=1} - l u_{l+}^{mj} \restriction_{r=1}. \tag{A.19}$$

The functions on the right-hand side belong to $\mathcal{E}_{l-1}^0$ and $\mathcal{E}_{l+1}^0$, respectively. Employing these relations, we can work out the action of the outgoing Green's function on $\mathcal{E}_l^1$.

The outgoing Green's function for frequency k, with $\Im k \geq 0$, is given by

$$g_k(\boldsymbol{x}, \boldsymbol{y}) = \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}. \tag{A.20}$$

If $r = |\boldsymbol{x}| > 1$ and $|\boldsymbol{y}| = 1$, then we can expand g as

$$g_k(\boldsymbol{x}, \boldsymbol{y}) = ik \sum_{l=0}^{\infty} j_l(k) h_l^{(1)}(kr) \boldsymbol{P}_l; \tag{A.21}$$

here $\boldsymbol{P}_l$ is the orthogonal projection onto $\mathcal{E}_l^0$, and

$$j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+1/2}(z) \quad \text{and} \quad h_l^{(1)}(z) = \sqrt{\frac{\pi}{2z}} H_{l+1/2}^{(1)}(z); \tag{A.22}$$

see [15]. When $k = 0$, formula (A.21) reduces to

$$g_0(\boldsymbol{x}, \boldsymbol{y}) = \sum_{l=0}^{\infty} \frac{\boldsymbol{P}_l}{(2l+1) r^{l+1}}. \tag{A.23}$$

We let

$$G_k f(\boldsymbol{x}) = \int_{S_1^2} g_k(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) dA(\boldsymbol{y}). \tag{A.24}$$

If $\boldsymbol{\alpha}$ is a 1-form on S_1^2, then it has a unique extension to $T\mathbb{R}^3 \restriction_{S_1^2}$ that annihilates ∂_r, which we denote by $\boldsymbol{\alpha} \cdot d\boldsymbol{x}$. The extended form has a well-defined representation along S_1^2 as

$$\boldsymbol{\alpha} \cdot d\boldsymbol{x} = \sum_{j=1}^{3} \alpha_j \, dx_j. \tag{A.25}$$

If we extend the coefficients to be homogeneous functions of degree 0, then $i_{\partial_r}\boldsymbol{\alpha} \cdot d\boldsymbol{x} = 0$ implies that

$$\sum_{j=1}^{3} x_j \, d\alpha_j = -\boldsymbol{\alpha} \cdot d\boldsymbol{x}, \tag{A.26}$$

which will prove useful below.

PROPOSITION A.2 *If $|\boldsymbol{y}| = 1$ and $r = |\boldsymbol{x}| > 1$, then, applied component-wise, the action of G_k is given by*

$$\begin{aligned}
&G_k Y_l^m = ikj_l(k)h_l^{(1)}(kr)Y_l^m, \\
&G_k[(\star_2 d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}] = ikj_l(k)h_l^{(1)}(kr)(\star_2 d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}, \\
&G_k\big[(d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}\big] = \\
&\quad ik\bigg[\left(\frac{dU_l^m}{2l+1}\right)\frac{\big[(l+1)j_{l-1}(k)h_{l-1}^{(1)}(kr) + lj_{l+1}(k)h_{l+1}^{(1)}(kr)\big]}{r^{l-1}} \\
&\qquad - U_l^m \, dr \frac{lj_{l+1}(k)h_{l+1}^{(1)}(kr)}{r^l}\bigg].
\end{aligned} \tag{A.27}$$

On the right-hand sides of (A.27), *Y_l^m is homogeneous of degree* 0, *as are the coefficients of $d_{S_1^2} Y_l^m \cdot d\boldsymbol{x}$ and $\star_2 d_{S_1^2} Y_l^m \cdot d\boldsymbol{x}$. As above, U_l^m is the homogeneous harmonic polynomial of degree l, defined by Y_l^m.*

Along the unit sphere the normal components are given by

$$\begin{aligned}
&i_{\partial_r} dG_k Y_l^m = ik^2 j_l(k)\partial_k h_l^{(1)}(k)Y_l^m, \\
&i_{\partial_r} G_k[(\star_2 d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}] = 0, \\
&i_{\partial_r} G_k[(d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}] = \\
&\quad ik\left(\frac{l(l+1)}{2l+1}\right)\big[j_{l-1}(k)h_{l-1}^{(1)}(k) - j_{l+1}(k)h_{l+1}^{(1)}(k)\big]Y_l^m.
\end{aligned} \tag{A.28}$$

Along the unit sphere the tangential components are given by

$$\begin{aligned}
&[dG_k Y_l^m]\restriction_{TS_1^2} = ikj_l(k)h_l^{(1)}(k)d_{S_1^2} Y_l^m, \\
&\big[G_k[(\star_2 d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}]\big]\restriction_{TS_1^2} = ikj_l(k)h_l^{(1)}(k) \star_2 d_{S_1^2} Y_l^m, \\
&\big[G_k[(d_{S_1^2} Y_l^m) \cdot d\boldsymbol{x}]\big]\restriction_{TS_1^2} = \\
&\quad \left(\frac{ik}{2l+1}\right)\big[(l+1)j_{l-1}(k)h_{l-1}^{(1)}(k) + lj_{l+1}(k)h_{l+1}^{(1)}(k)\big]d_{S_1^2} Y_l^m.
\end{aligned} \tag{A.29}$$

We also need to compute the effect of $\star_3 d$ on these eigenforms.

PROPOSITION A.3 *Along the unit sphere we have*

$$\begin{aligned} i_{\partial_r} \star_3 dG_k\big[(\star_2 d_{S_1^2} Y_l^m)\cdot d\boldsymbol{x}\big] &= -ikl(l+1)j_l(k)h_l^{(1)}(k)Y_l^m,\\ i_{\partial_r} \star_3 dG_k\big[(d_{S_1^2} Y_l^m)\cdot d\boldsymbol{x}\big] &= 0. \end{aligned} \tag{A.30}$$

Finally, we need to calculate the tangential components of these forms; the identity satisfied by a 2-form

$$\star_3\boldsymbol{\alpha}\restriction_{TS_1^2} = \star_2[i_{\partial_r}\boldsymbol{\alpha}\restriction_{TS_1^2}] \tag{A.31}$$

along with (A.26) facilitate these computations.

PROPOSITION A.4 *Along the unit sphere we have*

$$\begin{aligned} &\big[\star_3 dG_k\big[(\star_2 d_{S_1^2} Y_l^m)\cdot d\boldsymbol{x}\big]\big]\restriction_{TS_1^2} = -ikd_{s_1^2}Y_l^m j_l(k)\big[h_l^{(1)}(k) + k\partial_k h_l^{(1)}(k)\big],\\ &[\star_3 dG_k[(d_{S_1^2} Y_l^m)\cdot d\boldsymbol{x}]]\restriction_{TS_1^2} =\\ &\left(\frac{ik\star_2 d_{s_1^2}Y_l^m}{2l+1}\right)\big[l(l+2)j_{l+1}(k)h_{l+1}^{(1)}(k) - (l-1)(l+1)j_{l-1}(k)h_{l-1}^{(1)}(k)\\ &\qquad + k\big[(l+1)j_{l-1}(k)\partial_k h_{l-1}^{(1)}(k) + lj_{l+1}(k)\partial_k h_{l+1}^{(1)}(k)\big]\big]. \end{aligned} \tag{A.32}$$

Given (A.11), (A.17), (A.19), (A.18), (A.21), and (A.31) the formulæ in these propositions are elementary calculations, which follow from the fact that

$$\boldsymbol{P}_l\restriction_{\mathcal{E}_l^0} = I_{\mathcal{E}_l^0} \quad\text{and}\quad \boldsymbol{P}_l\restriction_{\mathcal{E}_k^0} = 0 \quad \text{if } k \neq l. \tag{A.33}$$

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