Smart Mater. Struct. **17** (2008) 015042 (9pp) doi:10.1088/0964-1726/17/1/015042

Numerical calculation of electromagnetic properties including chirality parameters for uniaxial bianisotropic media

Alireza V Amirkhizi and Sia Nemat-Nasser

Center of Excellence for Advanced Materials, Department of Mechanical and Aerospace Engineering, University of California, San Diego, USA

Received 13 December 2006
Published 17 January 2008
Online at stacks.iop.org/SMS/17/015042

Abstract
Through the use of conductive straight wires or coils the electromagnetic properties of a composite material can be modified. The asymmetric geometry of the coils creates an overall chiral response. The polarization vectors rotate as an electromagnetic wave travels through such a medium. To calculate the chirality of a medium prior to its manufacturing, we developed a method to extract all four electromagnetic material parameter tensors for a general uniaxial bianisotropic composite based on the numerical simulation of the electromagnetic fields. Our method uses appropriate line and surface field averages in a single unit cell of the periodic structure of the composite material. These overall field quantities have physical meaning only when the microscopic variation of the electromagnetic fields in the scale of the unit cell is not important, that is when the wavelength of interest is significantly larger than the maximum linear dimension of the unit cell. The overall constitutive relations of the periodic structure can then be obtained from the relations among the average quantities.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The electric permittivity of a composite material can be modified by embedding very thin straight wire conductors within the composite. The use of conductive coils makes it possible to use thicker wires and more robust manufacturing techniques to tune the corresponding dielectric properties (Amirkhizi *et al* 2003, Plaisted *et al* 2003, Nemat-Nasser *et al* 2006). The asymmetric geometry of the coils in this case creates an overall chiral response. The electric and magnetic field vectors rotate as a wave propagates through such a medium. This effect can be useful in certain applications but may create problems in other cases. Some ideas on how to eliminate the chirality effect have been discussed previously (Amirkhizi *et al* 2003). Ideally, one would want to calculate the chirality of a medium prior to its manufacturing. Here we propose a method to calculate all the material parameters, including chirality, for a general uniaxial bianisotropic structure, based on numerical simulation of wave propagation in periodic media. We start by integrating over appropriate lines and surfaces the field variables that are obtained using the results of a finite-element solver of Maxwell equations, following the method proposed by Pendry (1994, 1996). The average quantities derived in this manner are used as overall field quantities when the microscopic variation of the electromagnetic quantities in the scale of the unit cell is not important, that is when the wavelength of interest is longer than the maximum linear dimension of the unit cell by at least an order of magnitude. As suggested by the results presented in this paper, the values of the overall material parameters seem to be well behaved beyond this point. For example, in one case (uniform left-handed coils) the frequency band in this wavelength range (phase advance along a unit cell between 0 and $2\pi/10$) is 3.28–5.95 GHz, but for higher frequencies, the trend seems to follow the values of the parameters within this range. Furthermore, it has been established (Nemat-Nasser *et al* 2006) that the experimental results are compatible with assigning the overall material parameters to periodic structures even beyond the cutoff value. Therefore, it seems reasonable to extend the results beyond this limiting value, as we have done in the present work. The overall constitutive relations of the periodic structure are then obtained numerically from the

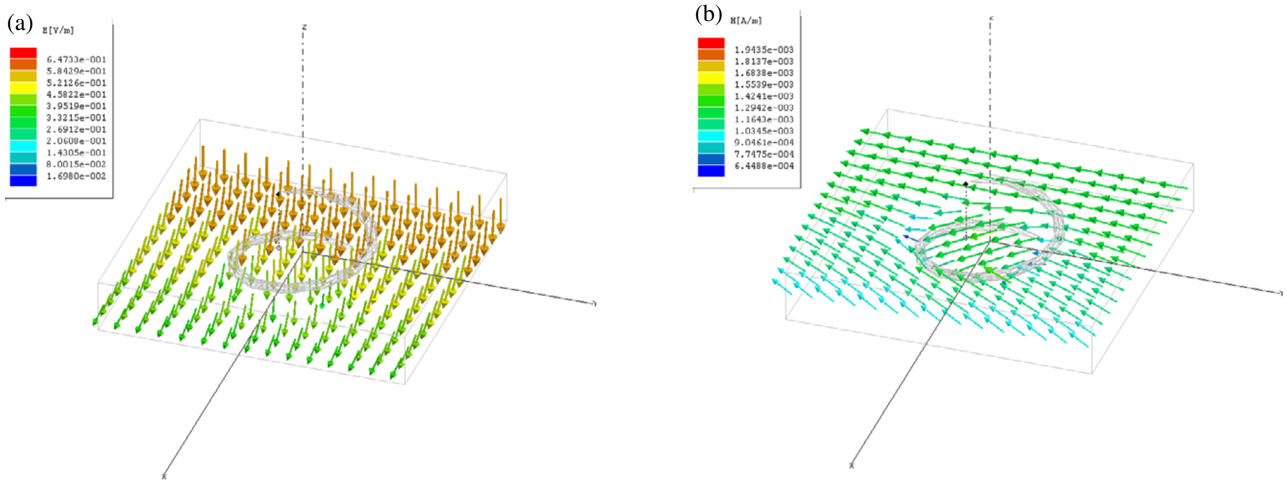

Figure 1. (Reproduced from Amirkhizi *et al* 2003.) Field patterns calculated for a unit cell of a coiled medium using Ansoft-HFSS: (a) the electric field **E**, and (b) the magnetic field **H**. The wave is propagating along the x-axis and the fields on the two yz-faces have 50° phase difference (the wavelength is 360/50 times the length of the cell in the x-direction). The electric field vector of the incoming wave on the far yz-face is at this time polarized parallel to the axis of the coil. However, the effect of the coil adds an out of phase normal component. Therefore the field vectors of both electric and magnetic fields are rotated as the wave travels through the medium.

relations among the average quantities. We apply this method to various chiral and non-chiral structures, and show that the results are in close agreement with previous work (Nemat-Nasser *et al* 2006).

2. Bianisotropic and chiral media

A chiral medium is characterized by the property that it is not identical with its own mirror image. The microstructure of a chiral medium has a left-handed or right-handed nature, and the left- or right-circularly polarized waves interact differently with such a medium. Optical activity which is observed in many molecules and is the basis of distinction between enantiomers (stereoisomers which are mirror images of each other) is used extensively in physical chemistry and optics. One manifestation of optical activity is circular dichroism, the difference in absorption of left- and right-circularly polarized light in chiral media. Optical rotary dispersion is another related phenomenon where the plane of the polarization of the transmitted plane wave through a slab of chiral material is rotated with respect to that of the incident wave; see figure 1. For a history of the discovery of optical activity see Lakhtakia *et al* (1989) and Lindell *et al* (1994). Lindman (1920, 1922) showed that similar phenomena can be created at microwave frequencies using artificial composites. He measured the rotation of the plane of polarization of a plane wave through a sample made of the same-handed copper helices embedded in foam. The term optical activity is not suitable for these experiments nor those done by Tinoco and Freeman (1957) which were performed at the X band. Instead the term electromagnetic activity or chirality is widely used in the literature. These works have a qualitative nature. A review of modern quantitative methods of measuring the chirality parameters at microwave frequencies can be found in chapter 10 of Chen *et al* (2004).

We are interested in chiro-electromagnetics or electromagnetic activity. A closely related phenomenon is the magneto-electric effect which was first postulated by Tellegen (1948) and systematically predicted by Landau and Lifshitz (1984). This phenomenon was experimentally confirmed for chromium oxide by Astrov (1961). The difference between the magneto-electric effect and the chirality or electromagnetic activity is clearly demonstrated when one studies the constitutive equations. In short, in a chiral material the electric field induces a magnetic polarization with which it has $\pi/2$ phase difference, and the magnetic field induces an electric polarization with similar phase difference. However in the magneto-electric effect, the induced magnetic polarization is in phase with the electric field and the induced electric polarization is in phase with the magnetic field.

The most general linear electromagnetic constitutive relation can be written as (Lindell *et al* 1994)

$$\mathbf{D} = \boldsymbol{\varepsilon} \cdot \mathbf{E} + \boldsymbol{\xi} \cdot \mathbf{H}, \tag{1}$$

$$\mathbf{B} = \boldsymbol{\zeta} \cdot \mathbf{E} + \boldsymbol{\mu} \cdot \mathbf{H}. \tag{2}$$

Here, $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$, $\boldsymbol{\xi}$, and $\boldsymbol{\zeta}$ are general second-order dyadics. The electrical permittivity, $\boldsymbol{\varepsilon}$, and the magnetic permeability, $\boldsymbol{\mu}$, are second-order tensors, while the chirality parameters $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are pseudo-tensors. For bi-isotropic media, the material parameters in (1), (2) are scalars and pseudo-scalars which are usually written as

$$\mathbf{D} = \varepsilon \mathbf{E} + (\chi - \mathrm{j}\kappa)\sqrt{\mu_0 \varepsilon_0}\mathbf{H}, \tag{3}$$

$$\mathbf{B} = (\chi + \mathrm{j}\kappa)\sqrt{\mu_0 \varepsilon_0}\mathbf{E} + \mu \mathbf{H}. \tag{4}$$

Here j is the imaginary unit. The distinction between the magneto-electric effect and chirality is clear here. When there is no chirality effect, $\kappa = 0$. When there is no magneto-electric effect, then $\chi = 0$. When the medium is lossless, ε, μ, χ, and

κ are real (Kong 1972). More generally, for a bianisotropic material characterized by (1), (2), the medium is lossless if

$$\boldsymbol{\varepsilon}^{+} = \boldsymbol{\varepsilon}, \tag{5}$$

$$\boldsymbol{\mu}^{+} = \boldsymbol{\mu}, \tag{6}$$

$$\boldsymbol{\zeta}^{+} = \boldsymbol{\xi}. \tag{7}$$

Here superscript $^{+}$ indicates the complex conjugate transpose. The medium is reciprocal if the following independent symmetry conditions are satisfied:

$$\boldsymbol{\varepsilon}^{\mathrm{T}} = \boldsymbol{\varepsilon}, \tag{8}$$

$$\boldsymbol{\mu}^{\mathrm{T}} = \boldsymbol{\mu}, \tag{9}$$

$$\boldsymbol{\zeta}^{\mathrm{T}} = -\boldsymbol{\xi}. \tag{10}$$

For a material that is both lossless and reciprocal, the chirality dyadic pseudo-tensors can be written as

$$\boldsymbol{\zeta} = -\boldsymbol{\xi} = \mathrm{j}\boldsymbol{\kappa}\sqrt{\mu_0\varepsilon_0}. \tag{11}$$

Here, $\boldsymbol{\kappa}$ is the real chirality dyadic. For bi-isotropic media, this would be a real pseudo-scalar.

The notation used here in not the only one in the literature for chiral media. Most notably, for bi-isotropic media, Born (1915) introduced the non-reciprocal form,

$$\mathbf{D} = \varepsilon(\mathbf{E} + \boldsymbol{\eta}\nabla\times\mathbf{E}), \tag{12}$$

$$\mathbf{B} = \mu\mathbf{H}. \tag{13}$$

This was later modified by Fedorov (1959) to

$$\mathbf{D} = \varepsilon(\mathbf{E} + \beta\nabla\times\mathbf{E}), \tag{14}$$

$$\mathbf{B} = \mu(\mathbf{H} + \beta\nabla\times H). \tag{15}$$

Note the non-local nature of this formulation. Condon (1937) used a formulation which is equivalent for harmonic fields:

$$\mathbf{D} = \varepsilon\mathbf{E} - \chi\frac{\partial\mathbf{H}}{\partial t}, \tag{16}$$

$$\mathbf{B} = \mu\mathbf{H} + \chi\frac{\partial\mathbf{E}}{\partial t}. \tag{17}$$

Other notable alternatives are those used by Post (1962),

$$\mathbf{D} = \varepsilon\mathbf{E} - \mathrm{j}\xi\mathbf{B} + \psi\mathbf{B}, \tag{18}$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} - \mathrm{j}\xi\mathbf{E} - \psi\mathbf{E}, \tag{19}$$

and by Kong (1972),

$$c\mathbf{D} = \mathbf{P}\cdot\mathbf{E} - c\mathbf{L}\cdot\mathbf{B}, \tag{20}$$

$$\mathbf{H} = \mathbf{M}\cdot\mathbf{E} + c\mathbf{Q}\cdot\mathbf{B}. \tag{21}$$

Here c is the speed of light in vacuum. The constitutive constants of these models are different but related. A collection of the relations among various representations can be found in Sihvola and Lindell (1991). We use the notation (1), (2) in the present work.

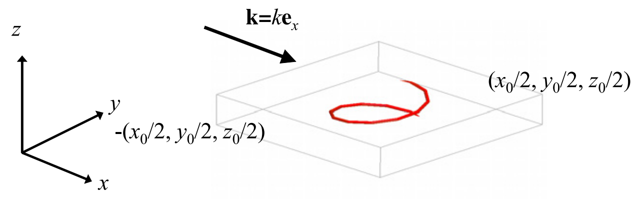

Figure 2. A rectangular cubic unit cell of a uniaxial periodic coil array.

3. Plane wave propagation in uniaxial media

The coiled medium considered here is a uniaxial medium; see figure 2. The z-axis, parallel to the axis of the coils, is the preferred axis in this case. A general uniaxial bianisotropic medium, with preferred z-axis, has the following electromagnetic dyadics:

$$\boldsymbol{\varepsilon} = \varepsilon_{\mathrm{t}}(\mathbf{1} - \mathbf{e}_z\otimes\mathbf{e}_z) + \varepsilon_z\mathbf{e}_z\otimes\mathbf{e}_z, \tag{22}$$

$$\boldsymbol{\mu} = \mu_{\mathrm{t}}(\mathbf{1} - \mathbf{e}_z\otimes\mathbf{e}_z) + \mu_z\mathbf{e}_z\otimes\mathbf{e}_z, \tag{23}$$

$$\boldsymbol{\xi} = \xi_{\mathrm{t}}(\mathbf{1} - \mathbf{e}_z\otimes\mathbf{e}_z) + \xi_z\mathbf{e}_z\otimes\mathbf{e}_z, \tag{24}$$

$$\boldsymbol{\zeta} = \zeta_{\mathrm{t}}(\mathbf{1} - \mathbf{e}_z\otimes\mathbf{e}_z) + \zeta_z\mathbf{e}_z\otimes\mathbf{e}_z. \tag{25}$$

An array of parallel perfectly conducting coils embedded in a lossless matrix constitutes a reciprocal and lossless composite. Therefore, equation (11) can be used. Furthermore, the chirality dyadics have the following simple form:

$$\boldsymbol{\zeta} = -\boldsymbol{\xi} = -\xi\mathbf{e}_z\otimes\mathbf{e}_z. \tag{26}$$

For more general uniaxial bianisotropic media, see Lindell *et al* (1993).

A harmonic plane wave traveling in such a medium can be represented by

$$\mathbf{F}(\mathbf{x}, t) = \mathbf{F}\mathrm{e}^{\mathrm{j}(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{27}$$

Here $\mathbf{F}$ is any of the four electromagnetic field vectors. The Maxwell equations in a medium without external sources then are

$$\mathbf{k}\cdot\mathbf{D} = 0, \tag{28}$$

$$\mathbf{k}\cdot\mathbf{B} = 0, \tag{29}$$

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}, \tag{30}$$

$$\mathbf{k}\times\mathbf{H} = -\omega\mathbf{D}. \tag{31}$$

The field vector $\mathbf{B}$ is normal to $\mathbf{E}$ as is $\mathbf{D}$ to $\mathbf{H}$. Substituting (22), (23), (26) into (28)–(31) gives two values of the wavenumber as functions of frequency and material properties (for the left- and right-circularly polarized waves):

$$k_{\pm} = \frac{\alpha}{\beta_{\pm}}, \tag{32}$$

$$\alpha = 2\omega^2\varepsilon_{\mathrm{t}}\mu_{\mathrm{t}}(\varepsilon_z\mu_z + \xi^2), \tag{33}$$

$$\beta_{\pm} = 2\cos^2\theta(\varepsilon_z\mu_z + \xi^2) + \sin^2\theta\Big(\varepsilon_{\mathrm{t}}\mu_z + \varepsilon_z\mu_{\mathrm{t}} \mp \sqrt{(\varepsilon_{\mathrm{t}}\mu_z - \varepsilon_z\mu_{\mathrm{t}})^2 - 4\varepsilon_{\mathrm{t}}\mu_{\mathrm{t}}\xi^2}\Big). \tag{34}$$

Here θ is the angle between the wavevector $\mathbf{k}$ and the z-axis, the axis of the coils. When the wave travels normal to the coils, (32)–(34) is simplified to

$$k_\pm = \frac{2\omega^2\varepsilon_t\mu_t(\varepsilon_z\mu_z + \xi^2)}{\varepsilon_t\mu_z + \varepsilon_z\mu_t \mp \sqrt{(\varepsilon_t\mu_z - \varepsilon_z\mu_t)^2 - 4\varepsilon_t\mu_t\xi^2}}. \tag{35}$$

Furthermore, for this special direction, one can show that the field vectors $\mathbf{E}$ and $\mathbf{H}$ are also normal to the wavevector $\mathbf{k}$; (see Lindell *et al* 1994, p 285). The electromagnetic wave traveling in this medium normal to the axis of the coils is a fully transverse wave, so

$$\mathbf{k}\cdot\mathbf{E} = 0, \tag{36}$$

$$\mathbf{k}\cdot\mathbf{H} = 0. \tag{37}$$

4. Average field quantities: definition based on line and surface integrals

In this section we expand the averaging method introduced by Pendry (1994, 1996) for chiral media. This method is based on the integral form of the Maxwell equations (without free external sources):

$$\oint_{\partial V} \mathbf{D}\cdot \mathrm{d}S = 0, \tag{38}$$

$$\oint_{\partial V} \mathbf{B}\cdot \mathrm{d}S = 0, \tag{39}$$

$$\oint_{\partial S} \mathbf{E}\cdot \mathrm{d}l = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot \mathrm{d}S, \tag{40}$$

$$\oint_{\partial S} \mathbf{H}\cdot \mathrm{d}l = \frac{\partial}{\partial t}\int_S \mathbf{D}\cdot \mathrm{d}S. \tag{41}$$

Here V and S are arbitrary volume and surface domains. Note that $\mathbf{E}$ and $\mathbf{H}$ only appear in line integrals, while $\mathbf{D}$ and $\mathbf{B}$ only appear in surface integrals. Therefore, if we can discretize the space in any manner and replace the local field quantities with appropriate averages (line integrals over element edges for $\mathbf{E}$ and $\mathbf{H}$, and surface integrals over element faces for $\mathbf{B}$ and $\mathbf{D}$), the macroscopic Maxwell equations are satisfied. This is the basis of the line and surface integral method for calculating the overall constitutive parameters. For a periodic composite, the natural discretization is the collection of unit cells.

Consider a rectangular cubic unit cell with faces at

$$x = \pm\frac{x_0}{2}, \tag{42}$$

$$y = \pm\frac{y_0}{2}, \tag{43}$$

$$z = \pm\frac{z_0}{2}. \tag{44}$$

When a wave of the form (27) travels in the positive x-direction in a periodic medium of this unit cell, the following boundary conditions hold:

$$\mathbf{f}\left(x, y = \frac{y_0}{2}, z\right) = \mathbf{f}\left(x, y = -\frac{y_0}{2}, z\right), \tag{45}$$

$$\mathbf{f}\left(x, y, z = \frac{z_0}{2}\right) = \mathbf{f}\left(x, y, z = -\frac{z_0}{2}\right), \tag{46}$$

$$\mathbf{f}\left(x = \frac{x_0}{2}, y, z\right) = \mathbf{f}\left(x = -\frac{x_0}{2}, y, z\right)\mathrm{e}^{\mathrm{j}\varphi}. \tag{47}$$

We denote the local microscopic field quantities by lower case letters $\mathbf{e}$, $\mathbf{b}$, $\mathbf{h}$, and $\mathbf{d}$, and let $\mathbf{f}$ stand for any of these four quantities. The phase advance is

$$\varphi = kx_0 = \frac{2\pi}{\lambda}x_0. \tag{48}$$

When the unit cell is heterogeneous, equation (27) is no longer applicable to the microscopic fields. However, the harmonic time dependence is still maintained. The electromagnetic fields can be written in the form

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x})\mathrm{e}^{-\mathrm{j}\omega t}. \tag{49}$$

The integral Maxwell equation (40) for the local fields on the face $z = z_0/2$ gives

$$\begin{aligned} &-\int_{(x,z)=(-\frac{x_0}{2},\frac{z_0}{2})} e_y\,\mathrm{d}y + \int_{(y,z)=(-\frac{y_0}{2},\frac{z_0}{2})} e_x\,\mathrm{d}x \\ &\quad + \int_{(x,z)=(\frac{x_0}{2},\frac{z_0}{2})} e_y\,\mathrm{d}y - \int_{(x,z)=(\frac{y_0}{2},\frac{z_0}{2})} e_x\,\mathrm{d}x \\ &= \mathrm{j}\omega\int_{z=\frac{z_0}{2}} b_z\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{50}$$

Using the periodic boundary conditions and defining

$$\bar{E}_y = \frac{1}{y_0}\int_{(x,z)=(-\frac{x_0}{2},\frac{z_0}{2})} e_y\,\mathrm{d}y, \tag{51}$$

$$\tilde{B}_z = \frac{1}{x_0 y_0}\int_{z=\frac{z_0}{2}} b_z\,\mathrm{d}x\,\mathrm{d}y, \tag{52}$$

equation (50) is simplified to

$$(\mathrm{e}^{\mathrm{j}kx_0} - 1)\bar{E}_y = \mathrm{j}\omega x_0\tilde{B}_z. \tag{53}$$

Further manipulation gives

$$k\bar{E}_y = \omega\frac{\mathrm{j}kx_0}{\mathrm{e}^{\mathrm{j}kx_0} - 1}\tilde{B}_z. \tag{54}$$

Define

$$\bar{B}_z = \frac{\mathrm{j}kx_0}{\mathrm{e}^{\mathrm{j}kx_0} - 1}\tilde{B}_z. \tag{55}$$

Then we have

$$k\bar{E}_y = \omega\bar{B}_z. \tag{56}$$

A similar procedure on the face $y = y_0/2$ gives

$$-k\bar{E}_z = \omega\bar{B}_y. \tag{57}$$

Equations (56) and (57) can be combined to obtain

$$(k\mathbf{e}_x)\times(\bar{E}_y\mathbf{e}_y + \bar{E}_z\mathbf{e}_z) = \omega(\bar{B}_y\mathbf{e}_y + \bar{B}_z\mathbf{e}_z). \tag{58}$$

Note that $\bar{E}_x$ can be defined in a similar manner. However, due to the periodicity of the medium, it is cancelled out of the Maxwell integral equations on any path over the surface of the unit cell. Furthermore, for any arbitrary $\bar{E}_x$, we have

$(k\mathbf{e}_x) \times (\bar{E}_x\mathbf{e}_x) = \mathbf{0}$. Therefore, exclusion of the longitudinal component of the electric field does not affect the final results. In fact, for a wave traveling normal to the axis of the coils in a uniaxial medium, we observed from numerical calculations that the value of $\bar{E}_x$, calculated based on numerical simulation, is almost zero. The overall electric field has no longitudinal component in a uniaxial bianisotropic medium for this wave, as is seen from equation (36).

The same exact procedure can be implemented for **D** and **H**. All four field quantities are obtained using the Ansoft-HFSS software and the necessary integrals are also calculated using built-in modules and the appropriate scripts that we have developed for this paper. This approach provides a rational basis for defining the average field quantities as follows:

$$\mathbf{E} = \left(\frac{1}{y_0}\int_{(x,z)=(-\frac{x_0}{2},\frac{z_0}{2})} e_y\,\mathrm{d}y\right)\mathbf{e}_y + \left(\frac{1}{z_0}\int_{(x,y)=(-\frac{x_0}{2},\frac{y_0}{2})} e_z\,\mathrm{d}z\right)\mathbf{e}_z, \tag{59}$$

$$\mathbf{B} = \frac{\mathrm{j}kx_0}{\mathrm{e}^{\mathrm{j}kx_0}-1}\left(\left(\frac{1}{x_0z_0}\int_{y=\frac{y_0}{2}} b_y\,\mathrm{d}y\right)\mathbf{e}_y + \left(\frac{1}{x_0y_0}\int_{z=\frac{z_0}{2}} b_z\,\mathrm{d}z\right)\mathbf{e}_z\right), \tag{60}$$

$$\mathbf{D} = \frac{\mathrm{j}kx_0}{\mathrm{e}^{\mathrm{j}kx_0}-1}\left(\left(\frac{1}{x_0z_0}\int_{y=\frac{y_0}{2}} d_y\mathrm{d}y\right)\mathbf{e}_y + \left(\frac{1}{x_0y_0}\int_{z=\frac{z_0}{2}} d_z\,\mathrm{d}z\right)\mathbf{e}_z\right), \tag{61}$$

$$\mathbf{H} = \left(\frac{1}{y_0}\int_{(x,z)=(-\frac{x_0}{2},\frac{z_0}{2})} h_y\,\mathrm{d}y\right)\mathbf{e}_y + \left(\frac{1}{z_0}\int_{(x,y)=(-\frac{x_0}{2},\frac{y_0}{2})} h_z\,\mathrm{d}z\right)\mathbf{e}_z. \tag{62}$$

A potential confusion may need some clarification at this point. First, the fields **E** and **H** are integrated on a front of the plane wave. Therefore their phase $\mathbf{k}\cdot\mathbf{x}$ is constant along the integration domain. However, the fields **B** and **D** are integrated over surfaces on which the phase is variable. It can be seen that the correction factor, $\mathrm{j}kx_0/(\mathrm{e}^{\mathrm{j}kx_0}-1)$, in (55) and (60), (61), once the equations are averaged, cancels out the phase difference between the surface integrals that give **B** and **D** and line integrals that give **E** and **H**. Note that in the effective medium limit where $x_0 \ll \lambda$, $\mathbf{k}\cdot\mathbf{x} \to 0$, and therefore $\mathrm{j}kx_0/(\mathrm{e}^{\mathrm{j}kx_0}-1) \to 1$. Also note that due to the periodicity, if we define the fields as integrals on the $y = -y_0/2$ and $z = -z_0/2$ faces and the edges on these faces instead of the positive faces, we still arrive at the same values. More interestingly, if the fields **E** and **H** are defined on the $x = x_0/2$ edges instead of the ones on the $x = -x_0/2$, the resulting vectors have a phase difference of kx_0, based on the definitions (59) and (62). The correction factor in (60), (61) in this case is $\mathrm{j}k(-x_0)/(\mathrm{e}^{\mathrm{j}k(-x_0)}-1)$. The ratio of the new correction to the one in (60), (61) is $\mathrm{e}^{\mathrm{j}kx_0}$, which amounts to the same phase difference, kx_0, in the definition of **E** and **H**. For the purpose of extracting the material properties, this extra phase difference is obviously inconsequential.

5. Extraction of the overall material properties

The constitutive equations for the parallel-coil uniaxial medium can be written as

$$D_z = \varepsilon_z E_z + \xi H_z, \tag{63}$$

$$B_z = \mu_z H_z - \xi E_z, \tag{64}$$

$$D_y = \varepsilon_\mathrm{t} E_y, \tag{65}$$

$$B_y = \mu_\mathrm{t} H_y. \tag{66}$$

Here, $\zeta = -\xi$ is deduced from the general reciprocity condition for the chirality dyadics, $\boldsymbol{\zeta} = -\boldsymbol{\xi}^\mathrm{T}$. Based on these equations, the values of transverse permittivity and permeability are easily calculated:

$$\varepsilon_\mathrm{t} = \frac{D_y}{E_y}, \tag{67}$$

$$\mu_\mathrm{t} = \frac{B_y}{H_y}. \tag{68}$$

Equations (63), (64) involve three remaining unknowns. We need one more equation. The resonant frequency of the unit cell with boundary conditions (45)–(48) can be found using a numerical solver; here we have used Ansoft-HFSS frequency-domain solver (Ansoft 2001). In this finite-element solver we can create the geometry of the unit cell and prescribe the boundary conditions (45)–(48). The resulting solution to the Maxwell equations then is a wave traveling along the x-axis in an infinite periodic medium of the input unit cell and the wavelength λ from equation (48). In addition to the field solutions, the finite-element solver also calculates the resonant frequency of this problem. Substitution of the wave number and the resonant frequency into the dispersion relation (35) gives the necessary third equation to solve for the three unknowns. We rewrite these equations as follows. First we normalize equations (63) and (64) with respect to the transverse permittivity and permeability:

$$\bar{\varepsilon}_z = \frac{\varepsilon_z}{\varepsilon_\mathrm{t}}, \tag{69}$$

$$\bar{\mu}_z = \frac{\mu_z}{\mu_\mathrm{t}}, \tag{70}$$

$$\bar{\xi} = \frac{\xi}{\sqrt{\mu_\mathrm{t}\varepsilon_\mathrm{t}}}. \tag{71}$$

From (63) and (64), we get

$$\bar{\varepsilon}_z = A - C\bar{\xi}, \tag{72}$$

$$\bar{\mu}_z = B + \frac{1}{C}\bar{\xi}. \tag{73}$$

Here, the parameters are defined as

$$A = \frac{D_z}{\varepsilon_\mathrm{t} E_z}, \tag{74}$$

$$B = \frac{B_z}{\mu_\mathrm{t} H_z}, \tag{75}$$

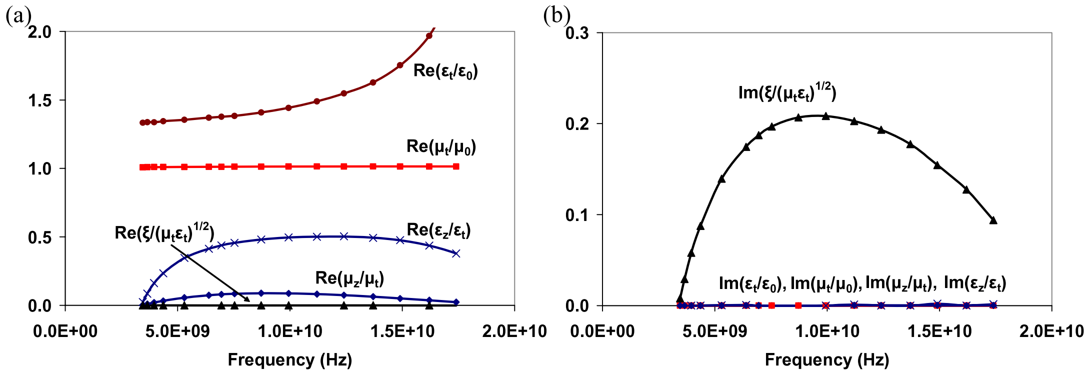

Figure 3. The overall material properties of the left-handed coil medium: the real part (a) and the imaginary part (b) of all five parameters.

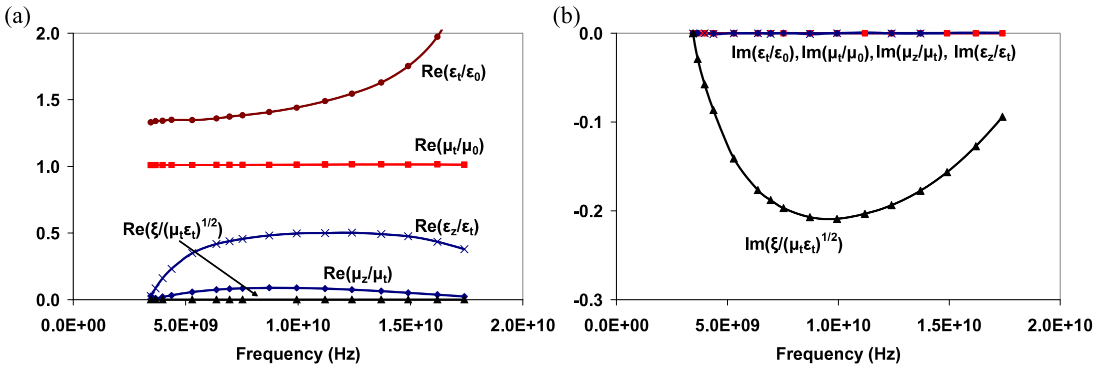

Figure 4. The overall material properties of the right-handed coil medium: the real part (a) and the imaginary part (b) of all five parameters.

$$C = \frac{H_z}{E_z}\sqrt{\frac{\mu_t}{\varepsilon_t}}. \tag{76}$$

We also define

$$D = \frac{k^2}{\omega^2 \varepsilon_t \mu_t}. \tag{77}$$

The dispersion relation (35) can now be written as

$$D = \frac{2(\bar{\varepsilon}_z \bar{\mu}_z + \bar{\xi}^2)}{\bar{\mu}_z + \bar{\varepsilon}_z \mp \sqrt{(\bar{\mu}_z - \bar{\varepsilon}_z)^2 - 4\bar{\xi}^2}}. \tag{78}$$

Substitution of (72), (73) in (78) gives a second-order equation for $\bar{\xi}$ (the apparent fourth-order terms are algebraically eliminated). The roots of this equation give two sets of possible parameters. The correct set can be found based on physical restrictions.

5.1. Numerical examples

In this section we present the results of several of the numerical examples that we worked out, using the procedure discussed above. In all cases, a basic unit cell with a left- or right-handed coil is used. The geometrical parameters of this unit cell are as follows. The spacing between the coils in the plane normal to their axis (cell width) is 6.35 mm. The coil pitch is 1 turn per 1.1 mm (cell height). The inner diameter of the coil is 2.6 mm and the wire thickness is 0.1 mm. The conductive coils are placed in vacuum.

In the first two examples, we consider a single left- or right-handed perfectly conductive coil. The unit cell of a right-handed coil is depicted in figure 2. We have used the Ansoft-HFSS (Ansoft 2001) frequency-domain solver to calculate the electric and magnetic fields within the unit cell for a given phase advance (see figure 1) as well as the resonant frequency. The software provides a calculator module that can integrate the field vectors on arbitrary line, surface, or volume domains. We have utilized it to evaluate the overall fields in (59)–(62). However, the software models a perfect conductor as a surface boundary condition and does not solve the Maxwell equations within the conductor. The perfect conductor is modeled by introducing necessary surface currents over its outside boundary. The integration scheme of the calculator module therefore neglects these significant currents. In our example, this affects the z-component of the electric displacement $\mathbf{D}$. Therefore, instead of the direct integral definition, we use the Maxwell equation (41) and calculate the average D_z from the integrated equation $\mathbf{k} \times \mathbf{H} = -\omega \mathbf{D}$:

$$D_z = -\frac{k}{\omega} H_y. \tag{79}$$

The real and imaginary parts of the overall material parameters for the left-handed coils are shown in figure 3. Note that, as expected, the overall composite is lossless; see equations (5)–(7). The material properties for the right-handed coil medium are shown in figure 4. In both these cases, the phase advance across a unit cell is $\pi/3$ at 8.72 GHz, and

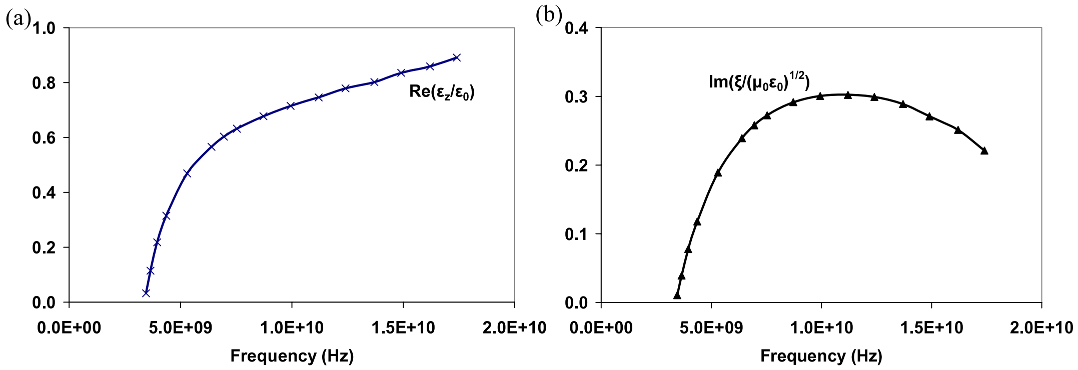

Figure 5. The overall material properties of the left-handed coil medium, normalized with respect to the permittivity of free space: the real part of the axial permittivity (a) and the imaginary part of the chirality parameter (b).

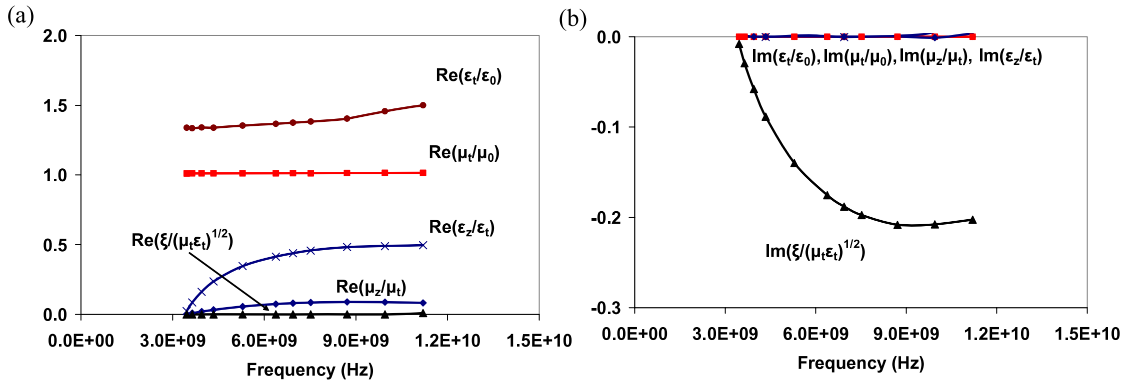

Figure 6. The overall material properties of the right-handed coil medium: the real part (a) and the imaginary part (b) of all five parameters. A unit cell consisting of two coils in the x-direction is used for this calculation. The agreement with the single unit cell (figure 4) is apparent.

$\pi/2$ at 12.4 GHz. This indicates that, at these frequencies, we are approaching the diffraction limit and that the overall properties may not be well defined beyond this range. For example, the deviation of the correction factor $\mathrm{j}kx_0/(\mathrm{e}^{\mathrm{j}kx_0}-1)$ in the definition of **B** and **D** from 1 becomes significant. Nevertheless, the method produces stable and reliable results.

As seen from figures 3 and 4, the only difference between the material properties of the left-handed and right-handed media is in the sign of $\mathrm{Im}(\xi)$. This is expected, as the permittivity and permeability tensors have to be invariant with respect to spatial reflection. The only components of the constitutive parameters that change with reflection are the chirality pseudo-tensors.

It must also be noted here that the apparent reduction in $\bar{\varepsilon}_z = \varepsilon_z/\varepsilon_t$ around 12 GHz is due to the increase in the transverse permittivity, ε_t. When the axial permittivity, ε_z, and the chirality parameter are normalized with respect to the permittivity of free space, the results shown in figure 5 are obtained.

The next example is used to check these results. We recalculated the material properties of the right-handed coil medium by combining two unit cells into one cell which now contains two parallel coils in the x-direction. The results are shown in figure 6 and are in very good agreement with the results given in figure 4.

Finally we solve a case where a unit cell contains a left-handed and a right-handed coil, parallel to one another, so that they cancel the chiral effect of each other, rendering the material non-chiral. Alternating the coils in either the x- or y-direction (or both) would cancel the effect of the polarization rotation. Here we consider the case with coils in the y-direction. The typical field solution for one eigenmode is shown in figure 7. The electric field is clearly polarized along the coil axis.

The y-component of the vectors **E** and **D**, and the z-component of **B** and **H** in the integral definitions (59)–(62) are zero, indicating that the composite is non-chiral. When this happens, the lengthy calculations involving the chirality constant are no longer necessary. One can use Pendry's anisotropic definitions:

$$\varepsilon_z = \frac{D_z}{E_z}, \tag{80}$$

$$\mu_t = \frac{B_y}{H_y}. \tag{81}$$

To calculate the transverse permittivity and axial permeability, we need to integrate the field solutions for a different eigenmode, for which the electric field is polarized normal to the coil axis. After proper integration, we use

$$\varepsilon_t = \frac{D_y}{E_y}, \tag{82}$$

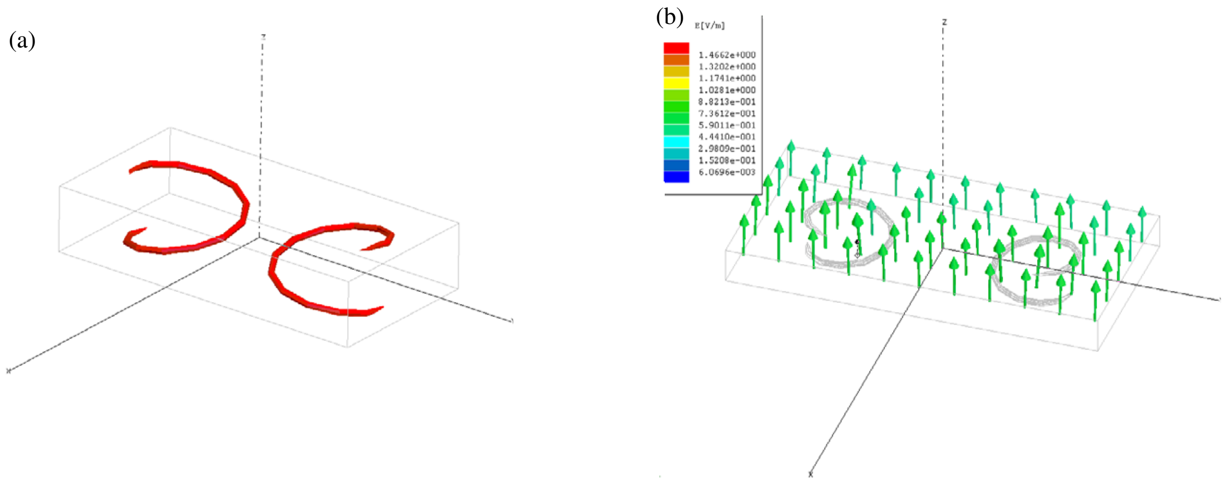

Figure 7. Unit cell with a left-handed and a right-handed coil in the y-direction (a), eliminating the effect of the field rotation (b). The linear polarization of the electric field parallel to the axis of the coils is maintained through the structure. The dispersion relation and plasmon frequency for the principal propagating modes remain essentially unaltered. However, the field eigenmodes are dramatically changed.

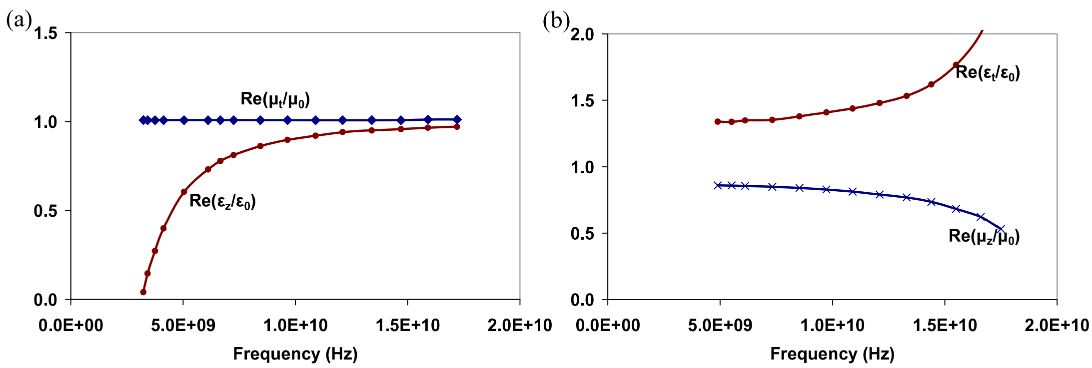

Figure 8. The overall material properties of a periodic array of alternating left-handed and right-handed coils. The unit cell and an eigenmode solution for axial polarization are shown in figure 6. (a) The axial permittivity and the transverse permeability related to the axial eigenmode. (b) The transverse permittivity and axial permeability related to the transverse eigenmode

$$\mu_z = \frac{B_z}{H_z}. \tag{83}$$

Note that in either of the two cases, since $\xi = 0$, the dispersion relation (77) will have the simpler form

$$D = \bar{\mu}_z \qquad \text{or} \qquad \bar{\varepsilon}_z. \tag{84}$$

For example, when the electric field is polarized along the coil axis, we obtain

$$D = \frac{k^2}{\omega^2 \mu_t \varepsilon_t} = \frac{\mu_z \varepsilon_t}{\mu_t \varepsilon_t} = \bar{\mu}_z. \tag{85}$$

The calculated constitutive parameters for the non-chiral alternating coil array are shown in figure 8. The dispersion relations (84) are satisfied for the two modes.

It is worth observing here that in all cases, the transverse permeability μ_t is very close to the permeability of free space and the axial permittivity ε_z shows a plasmonic response. In previous works (Amirkhizi *et al* 2003, Nemat-Nasser *et al* 2006) these assumptions were made based on physical arguments and were also shown to be compatible with the experimental results. The present work gives a numerical and theoretical verification of those assumptions.

An interesting outcome of the present examples is the small value calculated for the axial permeability μ_z. This value is consistent among all the chiral examples. On the other hand, when the medium is not chiral, although the overall axial permeability is less than that of free space, the reduction is far less pronounced; see figure 8.

6. Summary and conclusions

A method for the calculation of the overall electromagnetic material properties of periodic composites is presented here. We expand the method of Pendry (1994, 1996) for the cases where the material is uniaxially bianisotropic. The average **E** and **H** fields are calculated along lines in the axial and transverse directions with respect to the coils. The other two electromagnetic fields, **B** and **D**, are averaged appropriately (with a correction factor for the phase advance along the unit cell) on the surfaces normal and parallel to the common axis of the coils. These average fields are substituted in the constitutive equations, which are then solved together with the dispersion relation for the uniaxial medium to give the overall material constants of the periodic composites.

A number of numerical examples are worked out to demonstrate the effectiveness of the method. The method gives consistent results when it is applied to a single unit cell (figures 3 and 4) and a double unit cell (figure 6). Furthermore, comparison of the calculated material parameters of a left-handed coil medium (figure 3) with those of a right-handed coil (figure 4), shows that the only difference is the reversal of the sign of the chirality parameter. Finally, when an alternating left- and right-handed array (figure 7) is studied, it is revealed by calculation that the effect of chirality is canceled out.

Acknowledgments

This work was initially supported by DARPA through ARO DAAD19-00-1-0525 to the University of California, San Diego, and recently completed through partial support from AFOSR/MURI Grant FA9550-06-1-0337 to Kent State University, subaward 444286-PO61719 to University of California, San Diego.

References

Amirkhizi A V, Plaisted T A, Nemat-Nasser S C and Nemat-Nasser S 2003 Metallic coil–polymer braid composites: I. The numerical modeling and chirality *ICCM: 14th Ann. Int. Conf. on Composite Materials (San Diego, CA)* EM03-300

Ansoft 2001 *Ansfot HFSS 8.0 User Documentation* Ansoft Corporation, Pittsburgh, PA

Astrov D N 1961 Magnetoelectric effect in chromium oxide *Sov. Phys.—JETP* **13** 729–33

Born M 1915 Über die natürliche optische Aktivität von Flüssigkeiten und Gasen *Phys. Z.* **16** 251–8

Chen L F, Ong C K, Neo C P, Varadan V V and Varadan V K 2004 *Microwave Electronics: Measurement and Materials Characterization* (Chichester: Wiley)

Condon E U 1937 Theories of optical rotatory power *Rev. Mod. Phys.* **9** 432–57

Fedorov F I 1959 On the theory of optical activity of crystals. I. The law of conservation of energy and optical activity tensors *Opt. Spectrosc.* **6** 85–93

Kong J A 1972 Theorems of bianisotropic media *Proc. IEEE* **60** 1036–46

Lakhtakia A, Varadan V K and Varadan V V 1989 *Time-Harmonic Electromagnetic Fields in Chiral Media* (New York: Springer)

Landau L D and Lifshitz E M 1984 *Electrodynamics of Continuous Media* 2nd edn (Oxford: Butterworth-Heinemann)

Lindell I V, Viitanen A J and Koivisto P K 1993 Plane-wave propagation in a transversely bianisotropic uniaxial medium *Microw. Opt. Technol. Lett.* **6** 478–81

Lindell I V, Sihvola A H, Tretyakov S A and Viitanen A J 1994 *Electromagnetic Waves in Chiral and Bi-Isotropic Media* (Boston, MA: Artech House)

Lindman K F 1920 Über eine durch ein isotropes System von spiralförmigen Resonatoren erzeugte Rotationspolarisation der elektromagnetischen Wellen *Ann. Phys.* **63** 621–44

Lindman K F 1922 Über eine durch ein aktives Raumgitter erzeugte Rotationspolarisation der elektromagnetischen Wellen *Ann. Phys.* **69** 270–84

Nemat-Nasser S, Nemat-Nasser S C, Plaisted T A, Starr A and Vakil Amirkhizi A 2006 Multifunctional materials *Biomimetics: Biologically Inspired Technologies* ed Y Bar Cohen (Boca Raton, FL: CRC)

Pendry J B 1994 Photonic band structures *J. Mod. Opt.* **41** 209–29

Pendry J B 1996 Calculating photonic band structure *J. Phys.: Condens. Matter* **8** 1085–108

Plaisted T A, Amirkhizi A V, Arbelaez D, Nemat-Nasser S C and Nemat-Nasser S 2003 Metallic coil–polymer braid composites: II. Material processing and characterization *ICCM: 14th Ann. Int. Conf. on Composite Materials (San Diego, CA)* EM03-301

Post E J 1962 *Formal Structure of Electromagnetics: General Covariance and Electromagnetics* (Amsterdam: North-Holland)

Sihvola A H and Lindell I V 1991 Bi-isotropic constitutive relations *Microw. Opt. Technol. Lett.* **4** 295–7

Tellegen B D H 1948 The gyrator, a new electric network element *Philips Res. Rep.* **3** 81–101

Tinoco I and Freeman M P 1957 The optical activity of oriented copper helices: I. Experimental *J. Phys. Chem.* **61** 1196–200