# Opaque perfect lenses

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**Abstract**

The response of the "perfect lens", consisting of a slab of lossless material of thickness $d$ with $\varepsilon_s = \mu_s = -1$ at one frequency $\omega_0$ is investigated. It is shown that as time progresses the lens becomes increasingly opaque to any physical TM line dipole source located a distance $d_0 < d/2$ from the lens and which has been turned on at time $t = 0$. Here a physical source is defined as one which supplies a bounded amount of energy per unit time. In fact the lens cloaks the source so that it is not visible from behind the lens either. For sources which are turned on exponentially slowly there is an exact correspondence between the response of the perfect lens in the long time constant limit and the response of lossy lenses in the low loss limit. Contrary to the usual picture where the field intensity has a minimum at the front interface we find that the field diverges to infinity there in the long time constant limit.

*Key words:* Superresolution, Perfect lenses, Cloaking

## 1 Introduction

Recently there has been growing interest in superresolution, i.e. the fact that an image can be sharper than the wavelength of the radiation, which is in direct contrast to the proof of Abbe in 1873 that the resolution of a normal lens is at most about $\lambda/(2n)$ where $\lambda$ is the wavelength and $n$ is the refractive

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index. Although its significance was not recognized at the time, superresolution was implicitly discovered in 1994. Specifically it was found [1] that a coated cylinder with inner and outer radii $r_c$ and $r_s$ and having a real core dielectric constant $\varepsilon_c$, a shell dielectric constant $\varepsilon_s$ close to $-1$ (with a small positive imaginary part) and a matrix dielectric constant $\varepsilon_m = 1$ would have some rather strange properties in the quasistatic limit (where the free-space wavelength is infinitely long compared to the structure). In particular a line source aligned with the cylinder axis and positioned outside the cylinder radius at a radius $r_0$ with $r_* < r_0 < r_*^2/r_s$ where $r_* = r_s^2/r_c$ would have an arbitrarily sharp image positioned at a radius $r_*^2/r_0$ outside the coated cylinder. This image would only be apparent beyond the radius $r_*^2/r_0$; closer to the coated cylinder the potential was numerically found to exhibit enormous oscillations. The reason that one finds an image at this radius is that it was shown that the effect of the shell was to magnify the core, so it was equivalent to a solid cylinder of radius $r_*$. By the method of images in two-dimensional electrostatics the field outside the equivalent solid cylinder is that due to the actual source plus an image source at the radius $r_*^2/r_0$. However in contrast to electrostatics, the image source now lies in the physical region outside the coated cylinder. The paper [2] contains an in depth review of the results of the 1994 paper, correcting some minor errors.

Superresolution was rediscovered by Pendry [3], who realized its deep significance for imaging. He claimed that the Veselago lens, consisting of a slab of material having thickness $d$, relative electric permittivity $\varepsilon_s = -1$, relative magnetic permeability $\mu_s = -1$, and a refractive index of $-1$ would act as a superlens perfectly imaging the fields near the lens and shifting them by the distance $2d$. There were some flaws in his analysis. In particular a point source at a distance $d_0 < d$ from the lens, could not have an actual point source as its image, since this would imply a singularity in the fields there which cannot happen [4]. In fact there is no time harmonic solution in this case [5,6] since surface polaritons of vanishingly small wavelengths cause divergences [7]. While experiment has provided evidence for superresolution [8,9,10,11,12] to make theoretical sense of Pendry's claim one has to regularize the problem, say by making the slab lens slightly lossy or by switching on the source at finite time. A careful analysis of the lossy case was made in [13,14], and a rigorous mathematical proof of superlensing for quasistatic fields was given in [2] (see also [15] where a careful time harmonic analysis was given for real $\varepsilon_s$ and $\mu_s$ close, but not equal to $-1$). Both for the quasistatic case [2] and for the full time harmonic Maxwell equations [16,17] it was shown that contrary to the conventional explanation where the field intensity has a minimum at the front interface of the lens, the field actually diverges to infinity in two resonant layers of width $2(d-d_0)$, one centered on the front interface and one centered on the back interface. Indications of large fields in front of the lens [18,19,15,20] were followed by definitive numerical evidence of enormous fields [21]. When $d_0 < d/2$ the resonant layers interfere with the source. It was discovered [16] (following a suggestion of Alexei Efros that the energy absorbed by the lens may be infinite), that finite energy point or line sources or polarizable point or line dipoles less than a distance $d/2$ from the lens become cloaked, and are essentially invisible from outside the distance $d/2$ from the lens. Thus the Vesalago lens, in the limit as the loss tends to zero does not perfectly image physical sources that lie closer to the lens than a distance $d/2$.

The hope has persisted that a source turned on at time $t = 0$ would be perfectly imaged by a lossless Veselago lens (the perfect lens) as $t \to \infty$. This was first suggested by Gómez-Santos[22] and subsequently Yaghjian and Hansen[17] gave a detailed analysis. Both papers took into account the fact that due to dispersion $\mu_s(\omega)$ and $\varepsilon_s(\omega)$ can only equal $-1$ at one frequency $\omega_0$. At nearby frequencies one has

$$\begin{aligned} \varepsilon_s(\omega) &= -1 + a_\varepsilon(\omega - \omega_0) + O[(\omega - \omega_0)^2], \\ \mu_s(\omega) &= -1 + a_\mu(\omega - \omega_0) + O[(\omega - \omega_0)^2], \end{aligned} \tag{1.1}$$

where, due to causality, the dispersion coefficients (with $\varepsilon_s(\omega_0) = \mu_s(\omega_0) = -1$) necessarily satisfy the inequalities [17]

$$a_\varepsilon = \left.\frac{d\varepsilon_s}{d\omega}\right|_{\omega=\omega_0} \geq \frac{4}{\omega_0}, \qquad a_\mu = \left.\frac{d\mu_s}{d\omega}\right|_{\omega=\omega_0} \geq \frac{4}{\omega_0}, \tag{1.2}$$

which force them to be positive.

For simplicity it is assumed that the surrounding matrix material has $\mu_m = \varepsilon_m = 1$ for all frequencies. It was shown in these papers that the field at any given time would be finite except at the source. Also figure 1 in [22] shows the field has a local intensity minimum at the front interface and it was claimed in [17] that as $t \to \infty$ the field would diverge only in a single layer of width $2(d - d_0)$, centered on the back interface. However, here we will show that, again contrary to the conventional picture, the situation is precisely analogous to what occurs in a lossy lens as the loss goes to zero. The field also diverges to infinity in the layer of width $2(d - d_0)$ centered on the front interface, and as a consequence cloaking occurs when the source is less than a distance $d/2$ from the lens. The image of a constant energy source in this cloaking region becomes rapidly dimmer and dimmer as time increases. So instead of the lens being perfect, it is actually opaque to such sources, and cloaks them: not only is the source dim behind the lens, it is also dim in front of the lens. Essentially all of the energy produced by the source gets funnelled into the resonant regions which continually build up in intensity. Thus the claim [22] that "*even within the self-imposed idealizations of a lossless (for $\omega = \omega_0$) and purely homogeneous, left handed material, Pendry's perfect lens proposal is correct*" has to be qualified. It is *only true* for physical sources located further than a distance $d/2$ from the lens. For physical sources located less than a distance $d/2$ from the lens the image is completely different from what would appear if the lens were absent because the source interacts with the resonant fields in front of the lens.

## 2 Analysis

Simple energy considerations indicate that something strange must happen when $d_0 < d/2$. From equation (62) in [17] we see that a source of constant strength $E_0$ switched on at $t = 0$ creates an electric field which near the back interface scales approximately as

$$E \sim E_0 t^{1-d_0/d}. \tag{2.1}$$

The stored electrical energy $S_E(t)$ will scale as the square of this, and consequently the time derivative of the stored electrical energy will scale approximately as

$$\frac{dS_E}{dt} \sim E_0^2 t^{1-2d_0/d}, \tag{2.2}$$

which blows up to infinity as $t \to \infty$. If the source produces a bounded amount of energy per unit time we have a contradiction. The conclusion is that if the energy production rate of the source is bounded then necessarily $E_0$ must decrease to zero as $t \to \infty$. (If it approached any other equilibrium value then again we would have a contradiction). This sounds rather paradoxical but it could be explained if there was a resonant region in front of the lens, creating a sort of optical molasses, requiring ever increasing amounts of work to maintain the constant strength $E_0$.

Let us see that there is a resonant region in front of the lens through an adiabatic treatment of the problem. For simplicity we assume a TM line dipole source located along the $Z$-axis (which we capitalize to avoid confusion with $z = x + iy$) and that the slab faces are located at the planes $x = d_0$ and $x = d_0 + d$. Instead of assuming that the source is turned on sharply at $t = 0$ and thereafter remains constant we assume that it has been turned on exponentially slowly beginning in the infinite past. The source generates a field with the plane wave expansion

$$H_Z^{\mathrm{dip}}(x, y, t) = \int_{-\infty}^{\infty} dk_y \; a(k_y) e^{i(k_x x + k_y y - \omega t)} \quad \text{with} \quad k_x = \sqrt{\omega^2/c^2 - k_y^2}, \tag{2.3}$$

for $x > 0$, which interacts with the lens, where the coefficients $a(k_y)$ need to be determined and the square root in (2.3) is chosen so $\mathrm{Im}\, k_x > 0$ to ensure that the waves due to the source decay as $x$ increases. The frequency

$$\omega = \omega_0 + i/T \tag{2.4}$$

is complex and $T$ is a measure of the time the source has been "switched on" until time $t = 0$. It does not make sense to analyse this model in the limit as $t \to \infty$ since everything diverges exponentially in that limit. Rather we consider the model at time $t = 0$ at which point the source has been approximately constant for a very long period of time of the order of $T$. Thus investigating the asymptotic behavior as $T \to \infty$ at $t = 0$ in this model is analogous to investigating the asymptotic behavior as $t \to \infty$ of a constant amplitude source which has been switched on at time $t = 0$.

For a dipole line source we have

$$H_Z^{\mathrm{dip}}(x, y, t) = \frac{\pi\omega_0 e^{-i\omega t}}{2}\left(-k^{\mathrm{o}}\frac{\partial}{\partial x} + ik^{\mathrm{e}}\frac{\partial}{\partial y}\right) H_0^{(1)}\left((\omega/c)\sqrt{x^2+y^2}\right), \tag{2.5}$$

in which $H_0^{(1)}$ is a Hankel function of the first kind and $k^{\mathrm{e}}$ is the (possibly complex) strength at $t = 0$ of the dipole component which has an associated electric field with even symmetry about the $x$ axis and $k^{\mathrm{o}}$ is the (possibly complex) strength at $t = 0$ of the dipole component which has an associated electric field with odd symmetry about the $x$ axis: these dipole strengths have been normalized to agree with the definitions in [2] and [16]. By substituting the plane wave expansion [see formula (2.2.11) in [23]]

$$H_0^{(1)}\left((\omega/c)\sqrt{x^2+y^2}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk_y\, \frac{e^{i(k_x x + k_y y)}}{k_x}, \tag{2.6}$$

with

$$k_x = \sqrt{\omega^2/c^2 - k_y^2}, \tag{2.7}$$

in (2.6) we see that

$$a(k_y) = -\omega_0[k^{\mathrm{e}}(k_y/k_x) + ik^{\mathrm{o}}]/2. \tag{2.8}$$

We look for a particular solution of Maxwell's equations where all the fields, and not only the source, vary with time as $e^{-i\omega t}$ where $\omega$ is given by (2.4). This solution is obtained by substituting this complex value of $\omega$ into the time harmonic Maxwell's equations. Specifically with $\omega = \omega_0 + i/T$ and with the lens having the least possible dispersion, $\varepsilon_s$ and $\mu_s$ will according to (1.1) have the complex values

$$\varepsilon_s = -1 + ia_\varepsilon/T + O(1/T^2), \quad \mu_s = -1 + ia_\mu/T + O(1/T^2), \tag{2.9}$$

In other words, apart from the modulating factor of $e^{-i\omega t}$, the mathematical solution for the fields is exactly the same as for a lossy material with $\mu_s''$ and $\varepsilon_s''$ approximately proportional to $1/T$ for large $T$. A correspondence of this sort was noted before [17] but not fully exploited. By this argument it immediately follows that for fixed $k^{\mathrm{e}}$ and $k^{\mathrm{o}}$ the fields will diverge as $T \to \infty$ in *two* possibly overlapping layers of the same width $2(d - d_0)$ one centered on the back interface and one centered on the front interface. In particular, in front of the lens, with $2d_0 - d < x < d_0$, equations (4.18) and (4.19) of [16] imply

$$\begin{aligned} H_Z(x,y,t) \approx & \, H_Z^{\mathrm{dip}}(x,y,t) \\ & -\omega_0 e^{-i\omega t}\{[g^{\mathrm{e}}(z) - g^{\mathrm{e}}(\bar{z})]/2 + [g^{\mathrm{o}}(z) + g^{\mathrm{o}}(\bar{z})]/(2i)\}, \end{aligned} \tag{2.10}$$

where $z = x + iy$, $\bar{z} = x - iy$ and

$$g^{\mathrm{p}}(z) = -iqk^{\mathrm{p}}[a_\varepsilon/(2T)]^{(2d_0-d-z)/d}Q_0(2d - 2d_0 + z), \tag{2.11}$$

with

$$Q_0(b) = \frac{\pi}{2d\sin[\pi b/(2d)]}, \tag{2.12}$$

in which $q = 1$ for p=e and $q = -1$ for p=o. Thus we see that $g^{\mathrm{p}}(z)$ and hence $H_Z(x,y,t)$ diverges as $T \to \infty$ within a distance $d - d_0$ from the front of the lens. When $d_0 < d/2$ this resonant region interacts with the source creating the "optical molasses" that we mentioned. We have not done the computation, but presumably if one took $k^{\mathrm{o}} = 0$ and chose $k^{\mathrm{e}}$ to depend on $T$ in such a way that the source produces a given ($T$ independent) amount of energy at time $t = 0$ then one would find as $T \to \infty$ that the field would be localized and resonant in two layers of width $d$ which touch at the slab center. We remark that such field localization was found in the quasistatic case in the low loss limit [16] and also when two opposing sources are placed a distance $d/2$ behind and in front of the lens [24,25]

We only considered a particular solution to the equations. The general solution is of course the sum of a particular solution plus a solution to the homogeneous equations with no sources present, which we call a resonant solution. Since the lens is lossless, energy must be conserved and so a resonant solution which is zero and has zero total energy in the infinite past, must be zero for all time. Therefore the particular solution we considered is the only solution which satisfies the boundary condition of being zero in the infinite past.

No immediately apparent problems occur for line sources with $d_0$ between $d/2$ and $d$. While the stored electrical energy $S_E(t)$ in the resonant regions increases without bound, we see from (2.2) that the rate of increase diminishes with time. Similarly the rate of increase of magnetic energy diminishes with time. Therefore the image of such sources will get brighter and brighter as $t \to \infty$ approaching the same brightness as the original source without the lens present. However because the energy stored in the resonant regions is so large it may be the case that slight variations in the intensity of the source or slight non-linearities or slight inhomogeneities in the permeability and permittivity of the lens will scatter radiation and destroy the "perfect image". The spatial dispersion of the dielectric response of the slab will also limit resolution [26]. Finally we remark that we have assumed that the radiation coming from the source is coherent.

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## References

[1] N. A. Nicorovici, R. C. McPhedran, G. W. Milton, Optical and dielectric properties of partially resonant composites, Physical Review B (Solid State) 49 (12) (1994) 8479–8482.

[2] G. W. Milton, N.-A. P. Nicorovici, R. C. McPhedran, V. A. Podolskiy, A proof of superlensing in the quasistatic regime, and limitations of superlenses in this regime due to anomalous localized resonance, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 461 (2005) 3999–4034.

[3] J. B. Pendry, Negative refraction makes a perfect lens, Physical Review Letters 85 (2000) 3966–3969.

[4] D. Maystre, S. Enoch, Perfect lenses made with left-handed materials: Alice's mirror?, Journal of the Optical Society of America 21 (1) (2004) 122–131.

[5] N. Garcia, M. Nieto-Vesperinas, Left-handed materials do not make a perfect lens, Physical Review Letters 88 (2002) 207403.

[6] A. L. Pokrovsky, A. L. Efros, Diffraction in left-handed materials and theory of veselago lens ArXiv:cond-mat/0202078 v2 (2002).

[7] F. D. M. Haldane, Electromagnetic surface modes at interfaces with negative refractive index make a 'not-quite-perfect' lens ArXiv:cond-mat/0206420 v3 (2002).

[8] A. N. Lagarkov, V. N. Kissel, Near-perfect imaging in a focussing system based on a left-handed material plate, Physical Review Letters 92 (7) (2004) 077401.

[9] A. Grbic, G. V. Eleftheriades, Overcoming the diffraction limit with a planar left-handed transmission-line lens, Physical Review Letters 92 (11) (2004) 117403.

[10] N. Fang, H. Lee, C. Sun, X. Zhang, Sub-diffraction-limited optical imaging with a silver superlens, Science 308 (2005) 534–537.

[11] D. O. S. Melville, R. J. Blaikie, Super-resolution imaging through a planar silver layer, Optics Express 13 (6) (2005) 2127–2134.

[12] D. Korobkin, Y. Urzhumov, G. Shvets, Enhanced near-field resolution in mid-infrared using metamaterials, Journal of the Optical Society of America B 23 (3) (2006) 468–478.

[13] G. Shvets, Applications of surface plasmon and phonon polaritons to developing left-handed materials and nano-lithography, in: N. J. Halas (Ed.), Plasmonics: Metallic nanostructures and their optical properties, Vol. 5221 of Proceedings of SPIE, Society of Photo-Optical Instrumentation Engineers, Bellingham, WA,, 2003, pp. 124–132.

[14] V. A. Podolskiy, E. E. Narimanov, Near-sighted superlens, Optics Letters 30 (2005) 75–77.

[15] R. Merlin, Analytical solution of the almost-perfect-lens problem, Applied Physics Letters 84 (8) (2004) 1290–1292.

[16] G. W. Milton, N.-A. P. Nicorovici, On the cloaking effects associated with anomalous localized resonance, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences Published online May 3rd: doi:10.1098/rspa.2006.1715.

[17] A. D. Yaghjian, T. B. Hansen, Plane-wave solutions to frequency-domain and time-domain scattering from magnetodielectric slabs, Physical Review E (Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics) 73 (2006) 046608.

[18] X. S. Rao, C. K. Ong, Amplification of evanescent waves in a lossy left-handed material slab, Physical Review B (Solid State) 68 (2003) 113103.

[19] G. Shvets, Photonic approach to making a material with a negative index of refraction, Physical Review B (Solid State) 67 (2003) 035109.

[20] S. Guenneau, B. Gralak, J. B. Pendry, Perfect corner reflector, Optics Letters 30 (2005) 1204–1206.

[21] V. A. Podolskiy, N. A. Kuhta, G. W. Milton, Optimizing the superlens: manipulating geometry to enhance the resolution, Applied Physics Letters 87 (2005) 231113.

[22] G. Gómez-Santos, Universal features of the time evolution of evanescent modes in a left-handed perfect lens, Physical Review Letters 90 (2003) 077401.

[23] W. C. Chew, Waves and Fields in Inhomogeneous Media, IEEE Press Series on Electromagnetic Waves, IEEE Press, Piscataway, New Jersey, 1995.

[24] T. J. Cui, Q. Cheng, W. B. Lu, Q. Jiang, J. A. Kong, Localization of electromagnetic energy using a left-handed-medium slab, Physical Review B (Solid State) 71 (2005) 045114.

[25] A. D. Boardman, K. Marinov, Non-radiating and radiating configurations driven by left-handed metamaterials, Journal of the Optical Society of America B 23 (3) (2006) 543–552.

[26] I. A. Larkin, M. I. Stockman, Imperfect perfect lens, Nano Letters 5 (2) (2005) 339–343.