Negative Refractive Index in Left-Handed Materials

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The real part of the refractive index $n(\omega)$ of a nearly transparent and passive medium is usually taken to have only positive values. Through an analysis of a current source radiating into a 1D "left-handed" material (LHM)—where the permittivity and permeability are simultaneously less than zero—we determine the analytic structure of $n(\omega)$, demonstrating frequency regions where the sign of $\mathrm{Re}[n(\omega)]$ must, in fact, be negative. The regime of negative index, made relevant by a recent demonstration of an effective LHM, leads to unusual electromagnetic wave propagation and merits further exploration.

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In a paper published in 1968 [1], Veselago predicted that electromagnetic plane waves in a medium having simultaneously negative permittivity and permeability would propagate in a direction opposite to that of the flow of energy. This result follows not from the wave equation, which remains unchanged in the absence of sources, but rather from the individual Maxwell curl equations. The curl equation for the electric field provides an unambiguous "right-hand" (RH) rule between the directions of the electric field **E**, the magnetic induction **B**, and the direction of the propagation vector **k**. The direction of energy flow, however, given by $\mathbf{E} \times \mathbf{H}$, forms a right-handed system only when the permeability is greater than zero. Where the permeability is negative, the direction of propagation is reversed with respect to the direction of energy flow, the vectors **E**, **H**, and **k** forming a left-handed system; thus, Veselago referred to such materials as "left handed" (LH).

Veselago went on to argue, using steady-state solutions to Maxwell's equations, that a LH medium has a *negative* refractive index (n). While there are many examples of systems that can exhibit reversal of phase and group velocities with associated unusual wave propagation phenomena, negative group velocity bands in photonic crystals being an example, we show that the designation of negative refractive index is unique to LH systems. An isotropic negative index condition has the important property that it exactly reverses the propagation paths of rays within it; thus, LH materials have the potential to form highly efficient low reflectance surfaces by exactly canceling the scattering properties of other materials.

The absence of naturally occurring materials with negative μ made further discussion of LH media academic until recently, when a composite medium was demonstrated in which, over a finite frequency band, both the *effective permittivity* $\varepsilon(\omega)$ and the *effective permeability* $\mu(\omega)$ were shown to be simultaneously less than zero [2]. This physical medium was composed of distinct conducting elements, the size and spacing of which were on a scale much smaller than the wavelengths in the frequency range of interest. Thus, the composite medium could be considered homogeneous at the wavelengths under consideration. With this practical demonstration, it is now relevant to discuss in more detail the phenomena associated with wave propagation in LH materials, as both novel devices and interesting physics may result. Here we are are concerned with the interaction of waves with time dependent current sources in LH media.

The composite medium used in Ref. [2] made use of an array of metal posts to create a frequency region with $\varepsilon_{\mathrm{eff}} < 0$, interspersed with an array of split ring resonators (SRRs) having a frequency region with $\mu_{\mathrm{eff}} < 0$. The SRR medium and the wire array medium, both introduced by Pendry *et al.* [3–5], have been extensively studied previously. The thin wire medium can be described by the effective dielectric function

$$\varepsilon_{\mathrm{eff}}(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \tag{1}$$

with the plasma frequency ω_p related to the geometry of the wire array. The SRR medium can likewise be described by an effective frequency-dependent permeability, having the form

$$\mu_{\mathrm{eff}}(\omega) = 1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 - i\omega\Gamma}. \tag{2}$$

[We have chosen a slightly different form for the effective permeability than did Pendry *et al.* [5], to ensure that $\mu(\omega) \to 1$ for large frequency.]

Assuming that the wire and SRR arrays do not interact directly, the effective index of refraction of the composite medium can be taken as $n(\omega) = \sqrt{\varepsilon_{\mathrm{eff}}(\omega)\mu_{\mathrm{eff}}(\omega)}$, with the material constants given as in Eqs. (1) and (2). Thus, in the region $\omega_0 < \omega < \omega_b$ both μ_{eff} and $\varepsilon_{\mathrm{eff}}$ are simultaneously negative; the refractive index is real, and propagating modes exist.

The existence of negative material parameters is consistent with causality, which introduces the constraints [6]

$$\frac{d[\varepsilon(\omega)\omega]}{d\omega} > 1 \quad \text{and} \quad \frac{d[\mu(\omega)\omega]}{d\omega} > 1, \tag{3}$$

valid for nearly transparent media. Wave propagation and wave interaction with current sources in LH media are therefore necessarily complicated by the implicit frequency dependence of the material parameters, and even simple geometries can lead to mathematical or numerical complexity [7].

We are ultimately interested in the manner in which waves emanate from a current source and then propagate in a LH medium. The problem we consider here, shown in Fig. 1, consists of a sheet of current, infinite in the y and z directions and located at the position $x = x_0$, radiating into a LH medium. The magnitude of the current density is j_0, oriented along the z axis.

To illustrate the problem, we first solve for a harmonically time varying current density localized to the position x_0, assuming the resulting fields have the same time dependence. The 1D wave equation is thus

$$\frac{\partial^2 E(x)}{\partial x^2} + \frac{\Omega^2}{c^2}\mu\varepsilon E(x) = -\frac{4\pi i\Omega}{c^2}\mu j_0 \delta(x - x_0), \quad (4)$$

where Ω is the oscillation frequency of the source current. We further assume, in this preliminary example, that the material parameters are constant. Equation (4) then has solutions of the form

$$E(x,t) = -2\pi \frac{Z}{c} j_0 e^{i(nk|x-x_0|-\Omega t)}, \quad (5)$$

where $Z = \sqrt{\mu/\varepsilon} = \mu/n$ is the wave impedance.

When the product $\mu\varepsilon$ is real and positive, there are two types of solutions, corresponding to the sign of n being positive or negative. The $n < 0$ solution consists of plane waves propagating toward the source from plus and minus infinity, rather than plane waves propagating away from the source. Since such a solution would normally be rejected on the grounds of causality, we employ a general method that will distinguish the physical solution of Eq. (4) without presuming the sign of the index.

To identify the physical solution, we require that the source on average do positive work on the fields or that the quantity

$$P = \Omega W = -\frac{1}{2}\int_V j^* E(x,\Omega)\, dx = \pi \frac{\mu}{cn} j_0^2 \quad (6)$$

be greater than zero. W is the average work. In a RH medium, both μ and n are greater than zero, and the $n > 0$ solution is selected. In a LH medium, since μ is smaller than zero we conclude that the solution with $n < 0$ leads to the correct interpretation that the currents perform work on the fields. The latter solution is counterintuitive: Plane waves appear to propagate from plus and minus infinity toward the source, seemingly running "backwards in time" [8]. Yet, the work done by the source on the fields is positive, so clearly energy propagates outward from the source, in agreement with Veselago's assertions. The results are summarized in Fig. 2.

The general solution to the one-dimensional wave equation with an arbitrary current density can be written as the integral

$$E(x,t) = -\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx'\, z(x - x', t - t') \frac{j(x',t')}{c}, \quad (7)$$

where we define the *generalized impedance* as

$$z(x - x', t - t') = \int_{-\infty}^{\infty} d\omega \frac{\mu(\omega)}{n(\omega)} \times e^{i(\omega/c)[n(\omega)|x-x'|-c(t-t')]}. \quad (8)$$

We note that if the material constants are not frequency dependent, then Eq. (8) evaluates to

$$z(x - x', t - t') = 2\pi \frac{\mu}{n} \delta[n|x - x_0| - c(t - t')]. \quad (9)$$

For RH media, the delta function is nonzero when the two terms inside the parentheses are equal. For LH media,

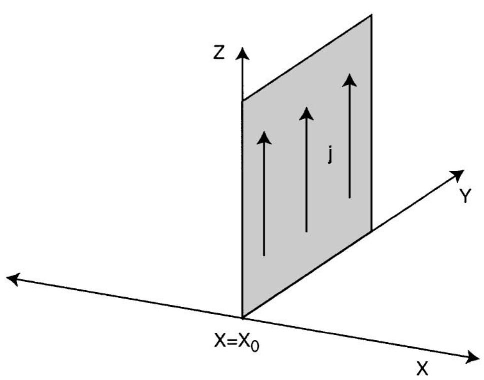

FIG. 1. A current sheet, located at position $x = x_0$, radiates into a left-handed medium. The current sheet is assumed to be uniform and infinite in the y and z directions.

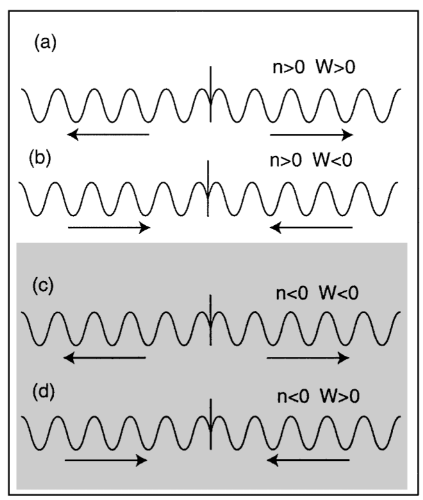

FIG. 2. Four possibilities for a source radiating into a 1D medium. (a) and (b) represent a right-handed medium (positive refractive index), while (c) and (d) represent a left-handed medium (negative refractive index). The arrows indicate the direction of the phase velocity. (a) and (d) represent cases where work is performed on the fields by the source. (b) and (c) represent cases where work is performed on the source by the fields. Note that the phase velocity is reversed in the left-handed medium (d).

however, there are no times $t > t'$ for which the delta function is nonzero, and thus there are no causal solutions for n negative and nondispersive. This result is not unexpected and restates that media with negative material constants must exhibit frequency dispersion.

In order to ascertain the general behavior of the radiation from a current sheet for arbitrary time dependent current pulses, we must evaluate the integral in Eq. (8). That solution, which will be presented elsewhere, requires a discussion of the analytic properties of the refractive index $n(\omega)$ and the generalized impedance $z(\omega)$, which we now consider. Both $n(\omega)$ and $z(\omega)$ are analytic in the upper half of the complex ω plane, as all singularities reside on (or below, when losses are included) the real axis. Furthermore, both $n(\omega)$ and $z(\omega)$ are double valued functions whose absolute values approach unity when $\omega \to \infty$, as the material constants ε and μ both approach unity.

When the imaginary part of the frequency is large and positive, the integrand in Eq. (8) approaches

$$I(\omega) \to e^{-(\omega/c)[n(\omega)|x-x'|-c(t-t')]}. \tag{10}$$

If a contour is chosen that runs along the real axis and closes in the upper half plane, the term in Eq. (10) vanishes along the closing contour when $t - t' < n(\omega)|x - x'|/c$; Eq. (8) can then be taken equivalent to the contour integral. The value of this integral is zero, as no singularities are enclosed, and we conclude that solutions not satisfying causality are equal to zero. Note that this requires $n(\omega) \to +1$ at $\omega = \infty$.

By requiring that the average work done on the field by each Fourier component of the current be positive, we can determine further conditions on the behavior of $n(\omega)$ and $z(\omega)$. The field at the same spatial location as the current source, in frequency space, is

$$E(\omega) = -z(\omega)\frac{j(\omega)}{c}, \tag{11}$$

where $z(\omega) = \mu(\omega)/n(\omega)$. Generalizing Eq. (6) for the rate of work done by the currents on the fields, we have

$$P(\omega) = \frac{1}{2} z(\omega)\frac{|j(\omega)|^2}{c}. \tag{12}$$

In order for the current to do positive work on the fields,

$$\mathrm{Re}[z(\omega)] \equiv z'(\omega) > 0, \tag{13}$$

as we restrict our discussion to passive materials. Here $z'(\omega)$ corresponds predominantly to radiation resistance.

For nearly transparent media, where $\mathrm{Im}[z(\omega)]$ can be neglected,

$$\begin{aligned}\frac{d(n\omega)}{d\omega} &= \frac{1}{2}\frac{d}{d\omega}\left(\omega\varepsilon z + \frac{\omega\mu}{z}\right) \\ &= \frac{1}{2}\left[z\frac{d(\omega\varepsilon)}{d\omega} + \frac{1}{z}\frac{d(\omega\mu)}{d\omega}\right].\end{aligned} \tag{14}$$

Applying Eq. (3) to Eq. (14), we see that $d(n\omega)/d\omega > (z + z^{-1})/2 > 1$, and we have a condition on the frequency dependence of the refractive index. Furthermore, the group velocity of a wave, defined as [6] $v_g = c/[d(n\omega)/d\omega]$, must therefore always be positive and less than c, in either LH or RH media.

While the real part of $z(\omega)$ is everywhere greater than or equal to zero, the same restriction does not occur for $n(\omega)$, except at $\omega \to \infty$. As $n(\omega)$ is analytic in the upper half plane, however, its value on the real axis is the boundary of an analytic function, and we can determine its form everywhere along the real axis by analytic continuation from the upper half plane.

To proceed further, it is convenient to choose a specific form for the material parameters; we use the forms presented in Eqs. (1) and (2). To make the analytic nature of the problem clearer, we write the index in a form that clearly displays the branch points, or

$$n(\omega) = \frac{1}{\omega}\sqrt{\frac{(\omega^2 - \omega_b^2)(\omega^2 - \omega_p^2)}{(\omega^2 - \omega_0^2)}}. \tag{15}$$

The branch points of $n(\omega)$ are shown in Fig. 3, where the white circles and $\times$'s indicate those arising from the poles and zeros in $n(\omega)$, respectively.

The distance between a frequency point in the complex plane infinitesimally close to the location of a singularity can be written in the form

$$\omega - \omega_s = re^{i\theta}, \tag{16}$$

where the coefficients r and θ are positive constants, $r \geq 0$, and ω_s is one of the singularities in Eq. (15). We imagine moving along the real axis in Fig. 3, starting from $\omega = +\infty$, where $n(\omega) \to +1$, and continuing toward the left. As we reach the first zero, $\omega_s = \omega_p$; we substitute Eq. (16) into Eq. (15), detouring around the zero into the upper half plane and back to the real axis. Since θ goes from $0 \to \pi$, $n(\omega)$ goes from positive real to positive imaginary. As we detour around each of the other zeros or poles shown in Fig. 3, we find that one of the two possible values for $n(\omega)$ is uniquely determined in each of the regions separated by singularities. Thus, for $\omega > \omega_p$ $\mathrm{Re}[n(\omega)] > 0$ and $\mathrm{Im}[n(\omega)] = 0$; for $\omega_b < \omega < \omega_p$ $\mathrm{Re}[n(\omega)] = 0$ and $\mathrm{Im}[n(\omega)] > 0$; for $\omega_0 < \omega < \omega_b$ $\mathrm{Re}[n(\omega)] < 0$ and $\mathrm{Im}[n(\omega)] = 0$; and for $0 < \omega < \omega_0$ $\mathrm{Re}[n(\omega)] = 0$ and $\mathrm{Im}[n(\omega)] > 0$. For $\omega < 0$, the signs on the imaginary terms are reversed, or $n(-\omega) = n(\omega^*)$.

The resulting behavior of plane waves in this composite material is now determined. We observe that plane waves

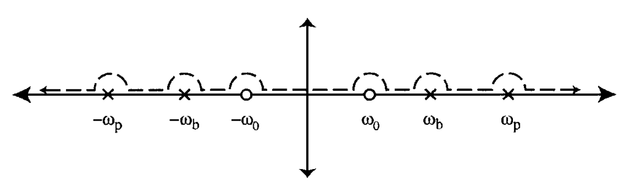

FIG. 3. The branch points ($\times$'s correspond to zeros; ○'s correspond to poles) and possible branch cuts associated with $n(\omega)$. The contour shown (dashed line) allows the continuation of $n(\omega)$ to the real axis.

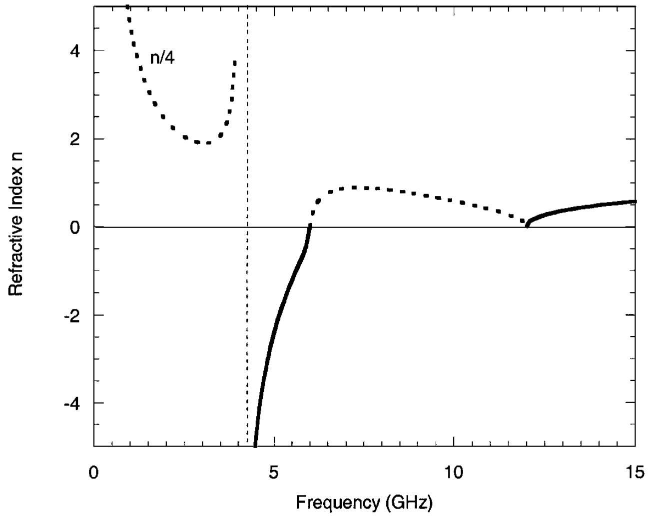

FIG. 4. Real (solid) and imaginary (dashed) branches of $n(\omega)$ versus frequency. The region where $n(\omega)$ is negative occurs when the permeability and permittivity are both negative. For this example, and to be somewhat compatible with the experimental values obtained in [2], we have chosen $\omega_p = 12.0$ GHz, $\omega_b = 6.0$ GHz, and $\omega_0 = 4.0$ GHz.

with frequencies above the plasma frequency are propagating. For frequencies below the plasma frequency, but not in the region between ω_0 and ω_b, modes are exponentially decaying. Finally, for frequencies that lie within the region between ω_0 and ω_b, modes are once again propagating but have a *negative* index of refraction. [Note that for arbitrary causal functional form, the analytic continuation procedure is guaranteed to lead to a positive imaginary result when either ε or μ is negative, as the corresponding waves must be evanescent. Hence, when both μ and ε are negative, $n = \sqrt{\varepsilon} \cdot \sqrt{\mu}$ must also be negative.] A plot of the refractive index is shown in Fig. 4, where solid curves represent the real branches and dashed curves represent the imaginary branches.

The presence of negative refractive index has been shown in steady-state problems to lead to unusual and unexplored phenomena in wave propagation [1]. We have presented here a formalism that can now be applied to time dependent currents and pulses in LH media, or combinations of LH and RH media.

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